Example 6.21
To prove that $f(x,y) = x^2 + y^2$ is convex, let $w_x, w_y, z_x, z_y \in \mathbb{R}$ and $\lambda \in [0,1]$ be given. We wish to show that

$$(\lambda w_x + (1-\lambda)z_x)^2 + (\lambda w_y + (1-\lambda)z_y)^2 \leq \lambda(w_x^2 + w_y^2) + (1-\lambda)(z_x^2 + z_y^2).$$

We start by nothing that because $0 \leq \lambda \leq 1$ $\lambda \geq \lambda^2$ and $(1-\lambda) \geq (1-\lambda)^2$ and also

$$\lambda - \lambda^2 = \lambda(1-\lambda) = (1-\lambda) - (1-\lambda)^2. \tag{1}$$

Therefore,

$$\begin{aligned}
\lambda w_x^2 + (1-\lambda)z_x^2 - \big(\lambda w_x + (1-\lambda)z_x\big)^2 &= \lambda w_x^2 + (1-\lambda)z_x^2 - \lambda^2 w_x - 2\lambda(1-\lambda)w_x z_x - (1-\lambda)^2 z_x \\
&= (\lambda - \lambda^2)w_x^2 + \big((1-\lambda) - (1-\lambda)^2\big)z_x^2 - 2\lambda(1-\lambda)w_x z_x \\
&= (\lambda - \lambda^2)w_x^2 + (\lambda - \lambda^2)z_x^2 - 2(\lambda - \lambda^2)w_x z_x \qquad \text{by (1).}
\end{aligned}$$

But then,

$$\lambda w_x^2 + (1-\lambda)z_x^2 - \big(\lambda w_x + (1-\lambda)z_x\big)^2 = (\lambda - \lambda^2)(w_x^2 - 2w_x z_x + z_x^2) = (\lambda - \lambda^2)(w_x - z_x)^2 \geq 0.$$

The same argument can be repeated for w_y and z_y. So

$$\begin{aligned}
\lambda f(w_x, w_y) + (1-\lambda)f(z_x, z_y) = \lambda(w_x^2 + w_y^2) + (1-\lambda)(z_x^2 + z_y^2) &\geq \big(\lambda w_x + (1-\lambda)z_x\big)^2 + \big(\lambda w_y + (1-\lambda)z_y\big)^2 \\
&= f(\lambda w_x + (1-\lambda)z_x, \lambda w_y + (1-\lambda)z_y).
\end{aligned}$$