

Lesson 2. An Illustrative Example

0 Warm up

On the axes on page 3, draw the following equations, and label their points of intersection.

$$20C + 40V = 260$$

$$4C + 2V = 32$$

$$4C + 5V = 40$$

1 Problem statement and data

Farmer Jones decides to supplement his income by baking and selling two types of cakes, chocolate and vanilla. Each chocolate cake sold gives a profit of \$3, and the profit on each vanilla cake sold is \$5. Each chocolate cake requires 20 minutes of baking time and uses 4 eggs and 4 pounds of flour, while each vanilla cake requires 40 minutes of baking time and uses 2 eggs and 5 pounds of flour. If Farmer Jones has available only 260 minutes of baking time, 32 eggs, and 40 pounds of flour, how many of each type of cake should be baked in order to maximize Farmer Jones's profit? (For now, assume all cakes baked are sold, and fractional cakes are OK.)

2 Model formulation

- An **optimization model** or **mathematical program** consists of:
 1. **Input parameters**: data that is given and fixed
 2. **Decision variables**: variables that represent decisions to be made
 3. **Constraints**
 - a. **Variable bounds**: specify the values for which the decision variables have meaning
 - b. **General constraints**: specify all other restrictions, requirements, and interactions that could limit the values of the decision variables
 4. **Objective function**: function of the decision variables to be maximized or minimized
- End up with something that looks like this:

(decision variable definitions)
maximize/minimize (objective function)
subject to (general constraints)
(variable bounds)

- Farmer Jones's optimization model:

C = number of chocolate cakes to bake

V = number of vanilla cakes to bake

$$\text{maximize } 3C + 5V \tag{1}$$

$$\text{subject to } 20C + 40V \leq 260 \tag{2}$$

$$4C + 2V \leq 32 \tag{3}$$

$$4C + 5V \leq 40 \tag{4}$$

$$C \geq 0, V \geq 0 \tag{5}$$

- Input parameters:

- (5):

- (2):

- (3):

- (4):

- (1):

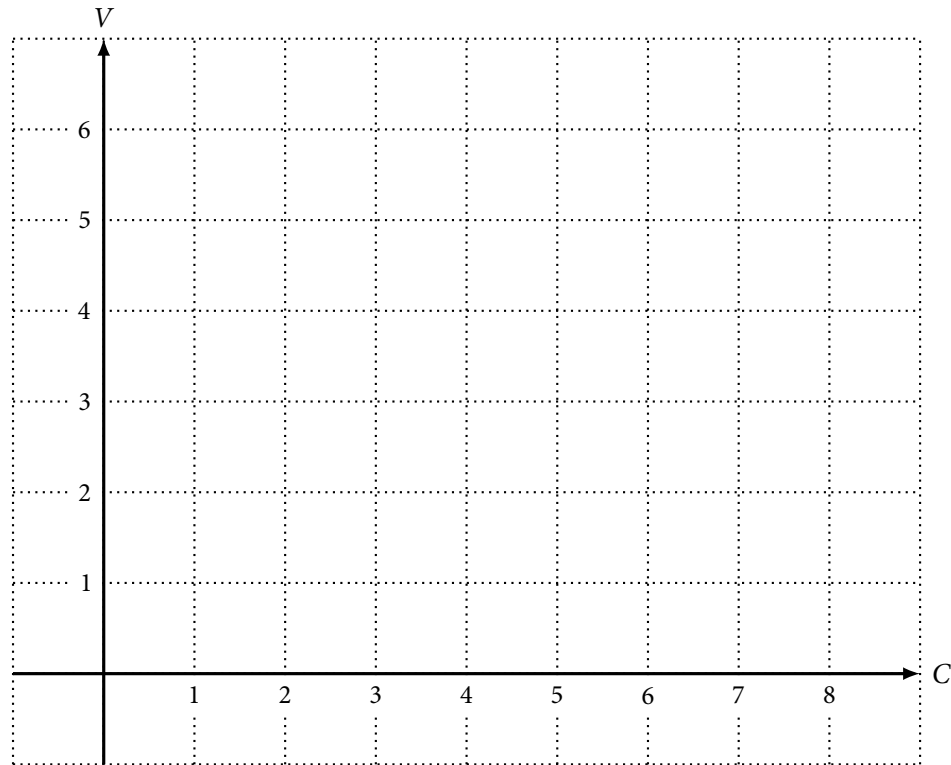
- Note that the constraints prevent the decision variables from taking on unallowable values, e.g. $C = 1000, V = 1000$

3 Basic terminology

- A **feasible solution** to an optimization model is a choice of values for the decision variables that satisfies all constraints
- The **feasible region** of an optimization model is the collection of all feasible solutions to the model
- The **value** of a feasible solution is its objective function value
- An **optimal solution** to an optimization model is a feasible solution whose value is as good as the value of all other feasible solutions

4 Solving the model

- Optimization models with 2 variables can be solved graphically
- The feasible region for Farmer Jones's optimization model:



- Any point in this shaded region represents a feasible solution
- How do we include the objective function? Look at **contour plots** of objective function
 - Lines/curves through points having equal objective function value
- For Farmer Jones's optimization model:
 - Test different objective function values by looking at lines of the form $3C + 5V = k$ for different values of k
 - Find the largest value of k such that the line $3C + 5V = k$ intersects the feasible region

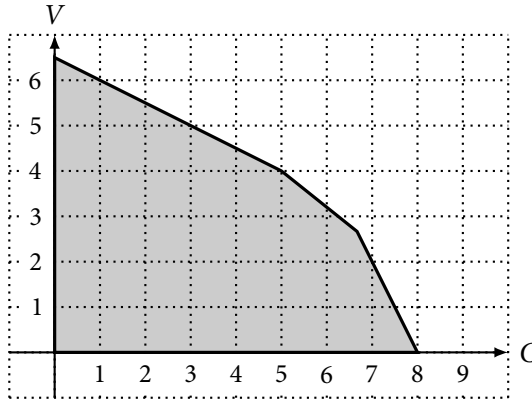
5 Outcomes of optimization models

- An optimization model may:
 1. have a **unique optimal solution**, or

2. have **multiple optimal solutions**

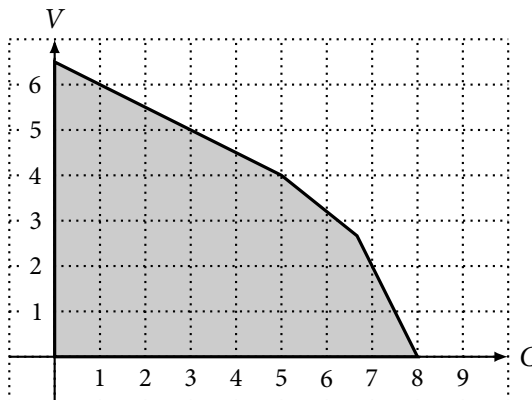
- e.g. What if the profit margin on vanilla cakes is \$6 instead?

Farmer Jones's objective function is then



3. be **infeasible**: no choice of decision variables satisfies all constraints

- e.g. What if the demands of Farmer Jones's neighbors dictate that he needs to bake at least 9 chocolate cakes? Then we need to add the constraint



4. be **unbounded**: for any feasible solution, there exists another feasible solution with a better value

- e.g. What if the circumstances have changed so that the feasible region of Farmer Jones's optimization model actually looks like this:

