

Lesson 8. Production Process Models, cont.

Problem. Alvin Fine produces three perfumes from raw material. Thirty thousand ounces of raw material is available. Each ounce of raw material can be transformed into 0.4 ounces of perfume 1, 0.3 ounces of perfume 2, and 0.2 ounces of perfume 3, while 0.1 ounces is lost as waste material. Each ounce of perfume 1 can be further processed into 0.6 ounces of perfume 2, 0.3 ounces of perfume 3, and 0.1 ounces of waste material. Alvin Fine has been contracted to produce at least 4000 ounces of perfume 1, 8000 ounces of perfume 2, and 10000 ounces of perfume 3. Because of its environmental initiatives it wishes to minimize waste material. Formulate a linear program that determines how much perfume to produce while minimizing waste.

Solution from class. In class on Friday, we proposed the following linear program for this problem:

Input parameters. Amount of available raw material, proportions of perfumes and waste generated from raw material, proportions of perfumes and waste generated from perfume 1, demands for perfumes.

Decision variables.

r = amount of raw material used

y = amount of perfume 1 further processed

w = total amount of waste generated

p_1 = amount of perfume 1 produced to meet demand

p_2 = amount of perfume 2 produced to meet demand

p_3 = amount of perfume 3 produced to meet demand

Objective function and constraints.

$$\text{minimize } w \quad (1a)$$

$$\text{subject to } r \leq 30000 \quad (1b)$$

$$y + p_1 = 0.4r \quad (1c)$$

$$p_2 = 0.3r + 0.6y \quad (1d)$$

$$p_3 = 0.2r + 0.3y \quad (1e)$$

$$w = 0.1r + 0.1y \quad (1f)$$

$$p_1 \geq 4000, p_2 \geq 8000, p_3 \geq 10000 \quad (1g)$$

$$r, w, y, p_1, p_2, p_3 \geq 0 \quad (1h)$$

The objective (1a) minimizes the total waste produced. Constraint (1b) ensures that Alvin Fine does not use more raw material than is available. Constraint (1c) ensures that the amount of perfume 1 transformed from raw material, $0.4r$, is equal to the amount of perfume 1 produced to meet demand, p_1 , plus the amount of perfume 1 further processed, y . Constraint (1d) ensures that the amount of perfume 2 produced to meet demand, p_2 , is equal to the amount of perfume 2 produced from both raw material, $0.3r$, and perfume 1, $0.6y$. Constraint (1e) does the same as constraint (1d), except for perfume 3. Constraint (1f) ensures that the total amount of waste generated, w , is equal to the amounts of waste generated from raw material, $0.1r$, and perfume 1, $0.1y$. Constraints (1g) ensure that demands for the three perfumes are met. Finally, constraints (1h) ensure that the amounts of perfumes and waste are nonnegative.

An alternate solution. Here is an alternate, but equivalent linear program:

Decision variables.

- r = amount of raw material used
- x_{r1} = amount of raw material transformed into perfume 1
- x_{r2} = amount of raw material transformed into perfume 2
- x_{r3} = amount of raw material transformed into perfume 3
- x_{rw} = amount of raw material transformed into waste
- x_{1f} = amount of perfume 1 further processed
- x_{12} = amount of perfume 1 transformed into perfume 2
- x_{13} = amount of perfume 1 transformed into perfume 3
- x_{1w} = amount of perfume 1 transformed into waste
- w = total amount of waste generated
- p_1 = amount of perfume 1 produced to meet demand
- p_2 = amount of perfume 2 produced to meet demand
- p_3 = amount of perfume 3 produced to meet demand

Objective function and constraints.

- minimize w (2a)
- subject to $r \leq 30000$ (2b)
- $x_{r1} = 0.4r, x_{r2} = 0.3r, x_{r3} = 0.2r, x_{rw} = 0.1r$ (2c)
- $x_{r1} = x_{1f} + p_1$ (2d)
- $x_{12} = 0.6x_{1f}, x_{13} = 0.3x_{1f}, x_{1w} = 0.1x_{1f}$ (2e)
- $p_2 = x_{r2} + x_{12}$ (2f)
- $p_3 = x_{r3} + x_{13}$ (2g)
- $w = x_{rw} + x_{1w}$ (2h)
- $p_1 \geq 4000, p_2 \geq 8000, p_3 \geq 10000$ (2i)
- $r, x_{r1}, x_{r2}, x_{r3}, x_{rw}, x_{1f}, x_{12}, x_{13}, x_{1w}, w, p_1, p_2, p_3 \geq 0$ (2j)

The objective (2a) minimizes the total waste produced. Constraint (2b) ensures that Alvin Fine does not use more raw material than is available. Constraints (2c) ensures that raw material is transformed into the three perfumes and waste in the correct proportions. Constraint (2d) ensures that the amount of perfume 1 transformed from raw material, x_{r1} , is equal to the amount of perfume 1 produced to meet demand, p_1 , plus the amount of perfume 1 further processed, x_{1f} . Constraints (2e) ensures that the perfume 1 that is further processed is transformed into the two other perfumes and waste in the correct proportions. Constraint (2f) ensures that the amount of perfume 2 produced to meet demand, p_2 , is equal to the amount of perfume 2 produced from both raw material, x_{r2} , and perfume 1, x_{12} . Constraint (2g) does the same as constraint (2f), except for perfume 3. Constraint (2h) ensures that the total amount of waste generated, w , is equal to the amounts of waste generated from raw material, x_{rw} , and perfume 1, x_{1w} . Constraints (2i) ensure that demands for the three perfumes are met. Finally, constraints (2j) ensure that the amounts of perfumes and waste are nonnegative.

Commentary. It's not too hard to see that these two linear programs are equivalent (this is left as an exercise for you). Although the second model has more variables, it is arguably easier to understand: the decision variables and constraints more closely reflect the different stages of Alvin Fine's production process.

Rule of thumb: an optimization model that is easier to understand is better.