# Lesson 17. Improving Search: Finding Better Solutions

**Solutions as vectors**

- A **feasible solution** to an optimization model is a choice of values for all decision variables that satisfies all constraints
- Suppose an optimization model has $n$ decision variables, $x_1, \dots, x_n$
- Easier to refer to solutions as **vectors**: for example, $\mathbf{x} = (x_1, \dots, x_n)$

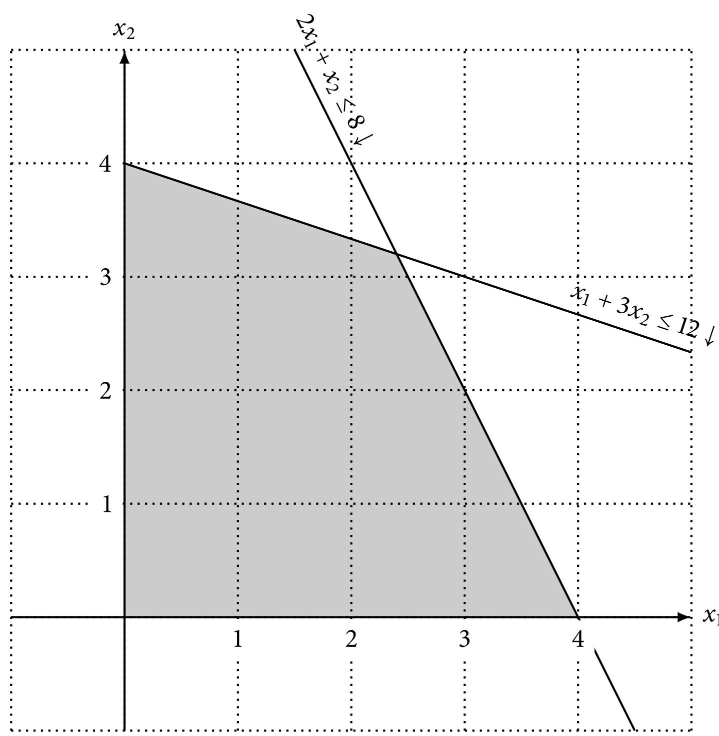
**A general optimization model with continuous variables**

- Let $\mathbf{x} = (x_1, \dots, x_n)$ be decision variables
- Let $f(\mathbf{x})$ and $g_i(\mathbf{x})$ for $i \in \{1, \dots, m\}$ be multivariable functions in $\mathbf{x}$, not necessarily linear
- Let $b_i$ for $i \in \{1, \dots, m\}$ be constant scalars

$$\begin{aligned} \text{maximize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} b_i \quad \text{for } i \in \{1, \dots, m\} \end{aligned}$$

**Example 1.**

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leq 12 && (1) \\ & 2x_1 + x_2 \leq 8 && (2) \\ & x_1 \geq 0 && (3) \\ & x_2 \geq 0 && (4) \end{aligned}$$



**Local and global optimal solutions**

- The $\varepsilon$-**neighborhood** of a solution $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ – denoted $N_\varepsilon(x)$ – is the set of all solutions $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ whose distance from $\mathbf{x}$ is less than $\varepsilon$.

$$N_\varepsilon(\mathbf{x}) = \left\{ \mathbf{y} \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \varepsilon \right\}$$

- A feasible solution $\mathbf{x}$ to an optimization model (with only continuous variables) is **locally optimal** if the value of $\mathbf{x}$ is better than or equal to the value of every other feasible solutions $\mathbf{y}$ in some $\varepsilon$-neighborhood of $\mathbf{x}$

- A feasible solution $\mathbf{x}$ to an optimization model is **globally optimal** if its value is better than or equal to the value of every other feasible solution
    - Otherwise known simply as an **optimal solution**

- Global optimal solutions are locally optimal, but not vice versa

- Hard to check for global optimality, easy to check for local optimality

**Improving search**

- General improving search algorithm

1 Find an initial feasible solution $\mathbf{x}^0$
2 Set $k = 0$
3 **while** $\mathbf{x}^k$ is not locally optimal **do**
4     Determine a new feasible solution $\mathbf{x}^{k+1}$ that improves the objective value at $\mathbf{x}^k$
5     Set $k = k + 1$
6 **end while**

- Generates sequence of feasible solutions $\mathbf{x}^0, \mathbf{x}^1, \dots$

- In general, improving search converges to a local optimal solution, not a global optimal solution

- Today: concentrate on Step 4 – finding better solutions

- How do we move from one solution to the next?

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \lambda \mathbf{d}$$

- For example:

**Improving directions**

- We want to choose $\mathbf{d}$ so that $\mathbf{x}^{k+1}$ has a better value than $\mathbf{x}^k$
- A vector $\mathbf{d}$ is an **improving direction** at the current solution $\mathbf{x}^k$ if the value of $\mathbf{x}^k + \lambda\mathbf{d}$ is better than the value of $\mathbf{x}^k$ for all positive $\lambda$ "close" to 0
- How do we find an improving direction?
- The **directional derivative** of $f$ in the direction $\mathbf{v}$ is

- Maximizing $f$: $\mathbf{d}$ is an improving direction at $\mathbf{x}^k$ if
- Minimizing $f$: $\mathbf{d}$ is an improving direction at $\mathbf{x}^k$ if
- In Example 1:

- For LPs: if $\mathbf{d}$ is an improving direction at $\mathbf{x}^k$, then the value of $\mathbf{x}^k + \lambda\mathbf{d}$ improves as $\lambda \to \infty$

**Step size**

- We have an improving direction $\mathbf{d}$ – now how far do we go?
- One idea: find maximum value of $\lambda$ so that $\mathbf{x}^k + \lambda\mathbf{d}$ is still feasible
- Graphically, we can eyeball this
- Algebraically, we can compute this – in Example 1:

**Feasible directions**

- Some improving directions don't lead to any new feasible solutions
- A constraint satisfied with equality at a feasible solution $\mathbf{x}$ is an **active constraint** at $\mathbf{x}$
  - Also: tight constraints, binding constraints
- A direction $\mathbf{d}$ is a **feasible direction** at $\mathbf{x}^k$ if $\mathbf{x}^k + \lambda\mathbf{d}$ is feasible for all positive $\lambda$ "close" to 0
- Again, graphically, we can eyeball this
- For LPs we have constraints of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b \quad \Leftrightarrow$$

- For LPs, a direction $\mathbf{d}$ is feasible at a feasible solution $\mathbf{x}$ if

$$\mathbf{a}^\top\mathbf{d} \begin{Bmatrix} \leq \\ \geq \\ = \end{Bmatrix} 0 \quad \text{for each active constraint of the form} \quad \mathbf{a}^\top\mathbf{x} \begin{Bmatrix} \leq \\ \geq \\ = \end{Bmatrix} b$$

- In Example 1:

**Detecting unboundedness**

- Suppose $\mathbf{d}$ is an improving direction at feasible solution $\mathbf{x}^k$ to an LP
- Suppose $\mathbf{x}^k + \lambda\mathbf{d}$ is feasible for all $\lambda \geq 0$
- What can you conclude?

**Summary**

- Step 4 boils down to finding an improving and feasible direction $\mathbf{d}$ and an accompanying step size $\lambda$
- We have conditions on whether a direction is improving and feasible
- We don't know how to find such directions... yet