Lesson 20. Geometry and Algebra of "Corner Points"

0 Warm up

Example 1. Consider the system of equations

$$\begin{aligned} 3x_1 + x_2 - 7x_3 &= 17 \\ x_1 + 5x_2 &= 1 \\ -2x_1 + 11x_3 &= -24 \end{aligned} \qquad (*)$$

Let

$$A = \begin{pmatrix} 3 & 1 & -7 \\ 1 & 5 & 0 \\ -2 & 0 & 11 \end{pmatrix}$$

We have that $\det(A) = 84$.

- Does $(*)$ have a unique solution, no solutions, or an infinite number of solutions?

- Are the row vectors of A linearly independent? How about the column vectors of A?

- What is the rank of A? Does A have full row rank?

1 Overview

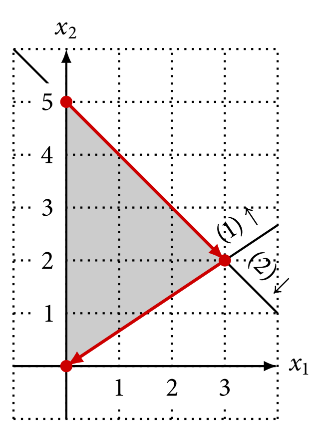
- Due to convexity, local optimal solutions of LPs are global optimal solutions

  ⇒ Improving search finds global optimal solutions of LPs

- Last time: improving search among "corner points" of the feasible region of an LP

- Today: how can we describe "corner points" of the feasible region of an LP?

- Coming next: for LPs, is there always an optimal solution that is a "corner point"?



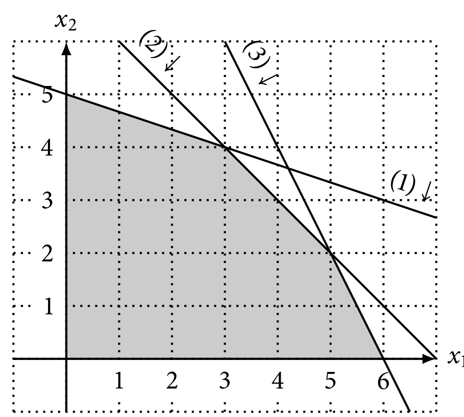
2 Polyhedra and extreme points

- A **polyhedron** is a set of vectors $\mathbf{x}$ that satisfy a finite collection of linear constraints (equalities and inequalities)
  - Also referred to as a **polyhedral set**
- In particular:

- Recall: the feasible region of an LP – a polyhedron – is a convex feasible region
- Given a convex feasible region S, a solution $\mathbf{x} \in S$ is an **extreme point** if there does not exist two distinct solutions $\mathbf{y}, \mathbf{z} \in S$ such that $\mathbf{x}$ is on the line segment joining $\mathbf{y}$ and $\mathbf{z}$
  - i.e. there does not exist $\lambda \in (0,1)$ such that $\mathbf{x} = \lambda \mathbf{y} + (1-\lambda)\mathbf{z}$

Example 2. Consider the polyhedron S and its graph below. What are the extreme points of S?

$$S = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{rl} x_1 + 3x_2 \leq 15 & (1) \\ x_1 + x_2 \leq 7 & (2) \\ 2x_1 + x_2 \leq 12 & (3) \\ x_1, x_2 \geq 0 & \end{array} \right\}$$



- "Corner points" of the feasible region of an LP $\Leftrightarrow$ extreme points

3 Basic solutions

- In Example 2, the polyhedron is described with 2 decision variables
- Each corner point / extreme point is the intersection of 2 lines
- Equivalently, each corner point / extreme point is active at 2 distinct constraints
- Is there a connection between the number of decision variables and the number of active constraints at a corner point / extreme point?
- Convention: all variables are on the LHS of constraints, all constants are on the RHS
- A collection of constraints defining a polyhedron are **linearly independent** if the LHS coefficient matrix these constraints has full row rank

Example 3. Consider the polyhedron S given in Example 2. Are constraints (1) and (3) linearly independent?

- Given a polyhedron S with n decision variables, $\mathbf{x}$ is a **basic solution** if
  (a) it satisfies all equality constraints
  (b) at least n constraints are active at $\mathbf{x}$ and are linearly independent

- $\mathbf{x}$ is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of S

Example 4. Consider the polyhedron S given in Example 2. Verify that $(3, 4)$ and $(21/5, 18/5)$ are basic solutions. Are these also basic feasible solutions?

4 Equivalence of extreme points and basic feasible solutions

- From our examples, it appears that for polyhedra, extreme points are the same as basic feasible solutions

Big Theorem. Suppose S is a polyhedron. Then $\mathbf{x}$ is an extreme point of S if and only if $\mathbf{x}$ is a basic feasible solution.

- See Rader p. 243 for a proof

- We use "extreme point" and "basic feasible solution" interchangeably

5 Food for thought

Problem 1. Does a polyhedron always have an extreme point? *Hint.* Consider the following polyhedron in $\mathbb{R}^2$: $S = \{(x_1, x_2) : x_1 + x_2 \geq 1\}$.

Problem 2 (also on today's homework)**.** Determine the extreme points of the following convex region:

$$\begin{aligned} -x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\leq 10 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$