Lesson 22. Linear Programs in Canonical Form

0 Warm up

Example 1. Let

$$A = \begin{pmatrix} 1 & 9 & 8 \\ 5 & 2 & 3 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Compute $A\mathbf{x}$.

1 Canonical form

- LPs in **canonical form** with decision variables $x_1, \dots, x_n$:

$$\begin{aligned} \text{minimize/maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i && \text{for } i \in \{1, \dots, m\} \\ & x_j \geq 0 && \text{for } j \in \{1, \dots, n\} \end{aligned}$$

- In vector-matrix notation with decision variable vector $\mathbf{x} = (x_1, \dots, x_n)$:

$$\begin{aligned} \text{minimize/maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

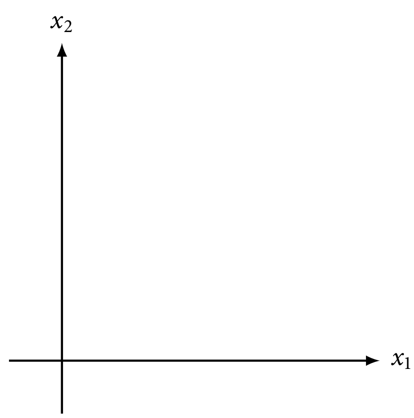
    - A has m rows and n columns, $\mathbf{b}$ has m components, and $\mathbf{c}$ and $\mathbf{x}$ has n components

- We typically assume that $m \leq n$, and $\text{rank}(A) = m$

Example 2. Identify $\mathbf{c}$, A, $\mathbf{b}$, and $\mathbf{x}$ in the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 3x + 4y - z \\ \text{subject to} \quad & 2x - 3y + z = 10 \\ & 7x + 2y - 8z = 5 \\ & x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

- A canonical form LP always has at least 1 extreme point (if it is feasible)
  - Intuition: if solutions in the feasible region must satisfy $\mathbf{x} \geq \mathbf{0}$, then the feasible region must be "pointed"



2 Converting any LP to an equivalent canonical form LP

- Inequalities → equalities
  - **Slack** and **surplus** variables "consume the difference" between the LHS and RHS
  - If constraint i is a $\leq$-constraint, add a slack variable s_i:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad \Rightarrow$$

  - If constraint i is a $\geq$-constraint, subtract a surplus variable s_i:

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad \Rightarrow$$

- Nonpositive variables → nonnegative variables
  - If $x_j \leq 0$, then introduce a new variable x'_j and substitute $x_j = -x'_j$ everywhere – in particular:

- Unrestricted ("free") variables → nonnegative variables
  - If x_j is unrestricted in sign, introduce 2 new nonnegative variables x_j^+, x_j^-
  - Substitute $x_j = x_j^+ - x_j^-$ everywhere
  - Why does this work?
    - Any real number can be expressed as the difference of two nonnegative numbers

Example 3. Convert the following LPs to canonical form.

$$\begin{array}{rl} \text{maximize} & 3x + 8y \\ \text{subject to} & x + 4y \leq 20 \\ & x + y \geq 9 \\ & x \geq 0,\ y \text{ free} \end{array} \qquad \begin{array}{rl} \text{minimize} & 5x_1 - 2x_2 + 9x_3 \\ \text{subject to} & 3x_1 + x_2 + 4x_3 = 8 \\ & 2x_1 + 7x_2 - 6x_3 \leq 4 \\ & x_1 \leq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

3 Next time...

- Basic solutions of canonical form LPs