

Lesson 27. Degeneracy, Convergence, Multiple Optimal Solutions

0 Warm up

Example 1. Suppose we are using the simplex method to solve the following canonical form LP:

$$\begin{aligned} &\text{maximize} && 10x + 3y \\ &\text{subject to} && x + y + s_1 = 4 \\ & && 5x + 2y + s_2 = 11 \\ & && y + s_3 = 4 \\ & && x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $\mathbf{x} = (x, y, s_1, s_2, s_3)$. Our current BFS is $\mathbf{x}^t = (0, 4, 0, 3, 0)$ with basis $\mathcal{B}^t = \{y, s_1, s_2\}$. The simplex directions are $\mathbf{d}^x = (1, 0, -1, -5, 0)$ and $\mathbf{d}^{s_3} = (0, -1, 1, 2, 1)$. Compute \mathbf{x}^{t+1} and \mathcal{B}^{t+1} .

- In the above example, the step size $\lambda_{\max} = 0$
- As a result, $\mathbf{x}^{t+1} = \mathbf{x}^t$: it looks like our solution didn't change!
- The basis did change, however: $\mathcal{B}^{t+1} \neq \mathcal{B}^t$
- Why did this happen?

1 Degeneracy

- A BFS \mathbf{x} of an LP with n decision variables is **degenerate** if there are more than n constraints active at \mathbf{x}
 - i.e. there are several collections of n linearly independent constraints that define the same \mathbf{x}

Example 2. Is \mathbf{x}^t in Example 1 degenerate? Why?

- In \mathbf{x}^t in Example 1, “too many” of the nonnegativity constraints are active
 - As a result, some of the basic variables are equal to zero
- Recall: a BFS of a canonical form LP with n decision variables and m equality constraints has
 - basic variables, potentially nonzero
 - nonbasic variables, always equal to 0
- Suppose \mathbf{x} is a degenerate BFS, with $n + k$ active constraints ($k \geq 1$)
- Then nonnegativity bounds must be active, which is larger than $n - m$
- Therefore: a BFS \mathbf{x} of a canonical form LP is degenerate if

- A degenerate BFS may correspond to several bases
- Every step of the simplex method
 - does not necessarily move to a geometrically adjacent extreme point
 - does move to an adjacent BFS (in particular, the bases differ by exactly 1 variable)
- At a degenerate BFS, the simplex method might “get stuck” for a few steps
 - Same BFS, different bases, different simplex directions
 - Zero-length moves: $\lambda_{\max} = 0$
- When $\lambda_{\max} = 0$, just proceed as usual
- Simplex computations will normally escape a sequence of zero-length moves and move away from the current BFS

2 Convergence

- In extreme cases, degeneracy can cause the simplex method to cycle over a set of bases that all represent the same extreme point
 - See Rader p. 291 for an example
- Can we guarantee that the simplex method terminates?
- Yes! Anticycling rules exist
- Easy anticycling rule: **Bland's rule**
 - Fix an ordering of the decision variables and rename them so that they have a common index
 - ◊ e.g. $(x, y, s_1, s_2, s_3) \rightarrow (x_1, x_2, x_3, x_4, x_5)$
 - Entering variable: choose nonbasic variable with smallest index among those corresponding to improving simplex directions
 - Leaving variable: choose basic variable with smallest index among those that define λ_{\max}

3 Multiple optimal solutions

- Suppose our current BFS is \mathbf{x}^t
- The change in objective function value from \mathbf{x}^t to $\mathbf{x}^t + \lambda \mathbf{d}$ ($\lambda \geq 0$) is

⇒ We can use reduced costs to compute changes in objective function

Example 3. Compute the change in objective value from \mathbf{x}^t to \mathbf{x}^{t+1} in Example 1.

- Suppose we solve a canonical form maximization LP with decision variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ using the simplex method, and end up with:

$$\begin{aligned} \mathbf{x}^t &= (0, 150, 0, 200, 50) & \mathcal{B}^t &= \{x_2, x_4, x_5\} \\ \mathbf{d}^{x_1} &= \left(1, -\frac{1}{2}, 0, -\frac{3}{2}, -\frac{1}{2}\right) & \mathbf{d}^{x_3} &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}\right) \\ \bar{c}_{x_1} &= 0 & \bar{c}_{x_3} &= -25 \end{aligned}$$

- Is \mathbf{x}^t optimal?

- Are there multiple optimal solutions?

- In general, if there is a reduced cost equal to 0 at an optimal solution, there may be other optimal solutions
 - The zero reduced cost must correspond to a simplex direction with $\lambda_{\max} > 0$