SA305 – Linear Programming Spring 2013
Asst. Prof. Nelson Uhan

Lesson 28. Finding an Initial BFS

1 Constructing the Phase I LP

- Idea for finding an initial BFS:
  - Construct an auxiliary LP based on the original canonical form LP – **the Phase I LP** – with an easy-to-find initial BFS
  - Solve the Phase I LP using the simplex method
  - The optimal solution to the Phase I LP will either
    - give an initial BFS for the original LP
    - prove that the original LP is infeasible

- How to construct the Phase I LP from the original canonical form LP:

  1. If necessary, multiply the equality constraints by −1 so that the RHS is nonnegative
  2. Add a nonnegative **artificial variable** to the LHS of each constraint
     (each constraint gets its own artificial variable)
  3. The objective is to minimize the sum of the artificial variables
  4. Compute the initial BFS for the Phase I LP by putting all artificial variables in the basis

Example 1. Construct the Phase I LP from the following canonical form LP.

$$\begin{aligned}
\text{maximize} \quad & 4x_1 + 5x_2 - 9x_3 \\
\text{subject to} \quad & 8x_1 - x_2 + x_3 = 4 \\
& x_1 + 4x_2 - 7x_3 = -22 \\
& x_1, \quad x_2, \quad x_3 \geq 0
\end{aligned}$$

2 How does the Phase I LP work?

- Let's consider the Phase I LP we wrote in Example 1
- The Phase I LP can't be unbounded, because
- It can't be infeasible either (we have a BFS!)
- Therefore, the Phase I LP must have an optimal solution
- Let $(x_1^*, x_2^*, x_3^*, a_1^*, a_2^*)$ be an optimal BFS to the Phase I LP
- **Case 1.** The optimal value of the Phase I LP is strictly greater than 0: $a_1^* + a_2^* > 0$
- **Case 2.** The optimal value of the Phase I LP is equal to 0: $a_1^* + a_2^* = 0$
- This reasoning applies more generally

3 Putting it all together: The Two-Phase Simplex Method

Step 1: Phase I. Construct Phase I LP and easy-to-find initial BFS. Use the simplex method to solve the Phase I LP.

Step 2: Infeasibility. If the optimal value of the Phase I LP is

- $> 0 \Rightarrow$ stop; original LP is infeasible.
- $= 0 \Rightarrow$ identify initial BFS for original LP.

Step 3: Phase II. Use the simplex method to solve the original LP, using the initial BFS identified in Step 2.

4 Possible outcomes of LPs

- When do we detect if an LP:

is infeasible?	is unbounded?	has an optimal solution?