

Lesson 30. Weak and Strong Duality

0 Warm up

Example 1. Consider the following LP and its equivalent canonical form LP:

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 8 \\ & 3x_1 + x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 + s_1 = 8 \\ & 3x_1 + x_2 + s_2 = 9 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

Define the decision variable vector $\mathbf{x} = (x_1, x_2, s_1, s_2)$. Solve the canonical form LP using the simplex method, with initial BFS $\mathbf{x}^0 = (3, 0, 5, 0)$ with basis $\mathcal{B}^0 = \{x_1, s_1\}$.

Example 2. Consider the following LP and its equivalent canonical form LP:

$$\begin{array}{ll} \text{minimize} & 8y_1 + 9y_2 \\ \text{subject to} & y_1 + 3y_2 \geq 2 \\ & 2y_1 + y_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & 8y_1 + 9y_2 \\ \text{subject to} & y_1 + 3y_2 - s_1 = 2 \\ & 2y_1 + y_2 - s_2 = 1 \\ & y_1, y_2, s_1, s_2 \geq 0 \end{array}$$

Define the decision variable vector $\mathbf{y} = (y_1, y_2, s_1, s_2)$. Solve the canonical form LP using the simplex method, with initial BFS $\mathbf{y}^0 = (0, 1, 1, 0)$ with basis $\mathcal{B}^0 = \{y_2, s_1\}$.

1 Weak duality

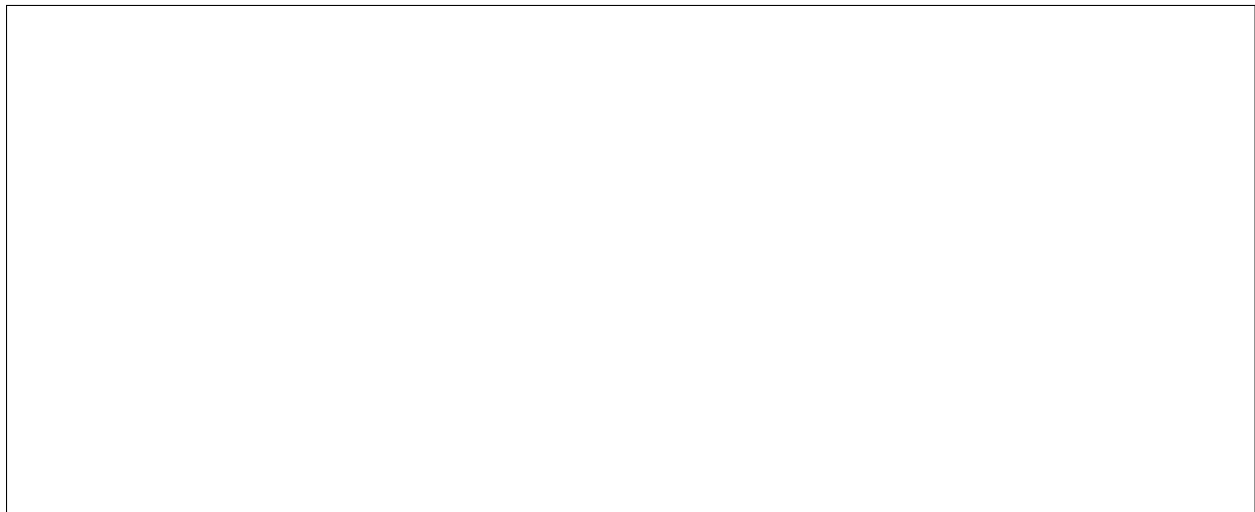
- Consider the following primal-dual pair of LPs

$$\begin{array}{ll} \text{[P]} & \text{maximize } \mathbf{c}^\top \mathbf{x} \\ & \text{subject to } A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{ll} \text{[D]} & \text{minimize } \mathbf{b}^\top \mathbf{y} \\ & \text{subject to } A^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

- Remember we constructed the dual in such a way that the multipliers \mathbf{y} give us an upper bound on the optimal value of [P]

Weak Duality Theorem. Let \mathbf{x}^* be a feasible solution to [P], and let \mathbf{y}^* be a feasible solution to [D]. Then

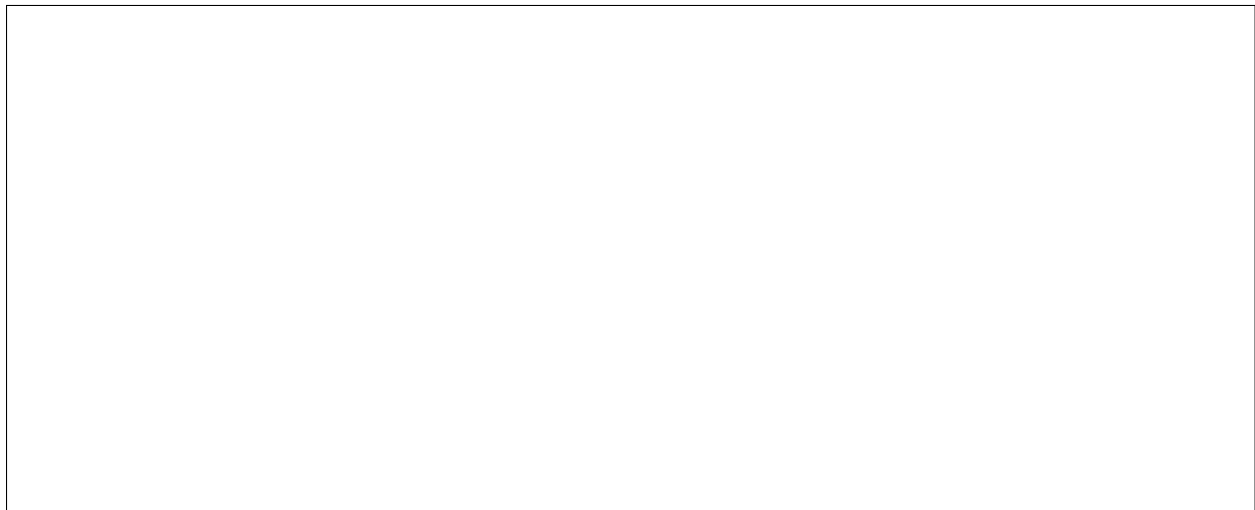
$$\mathbf{c}^\top \mathbf{x}^* \leq \mathbf{b}^\top \mathbf{y}^*$$



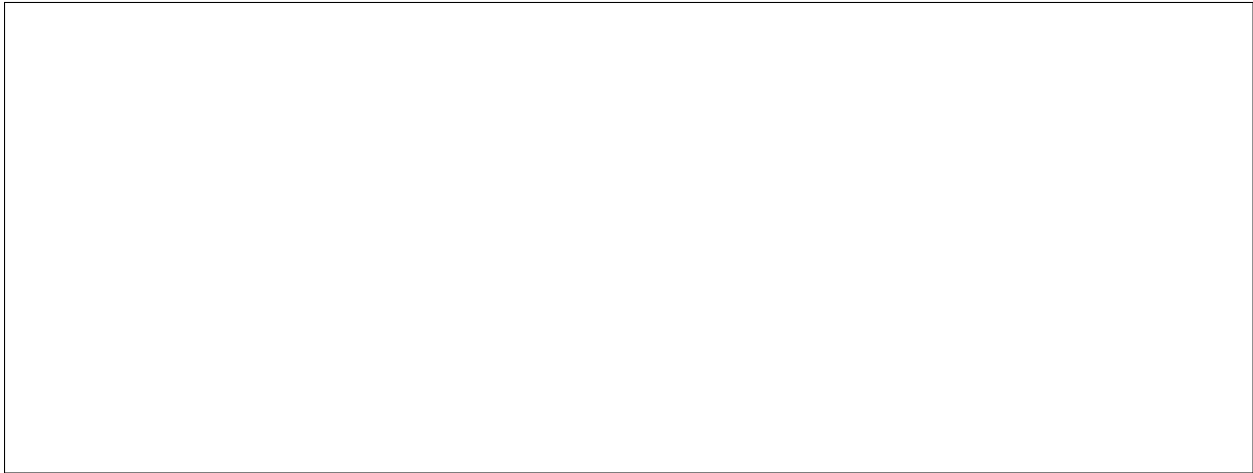
Corollary 1. If \mathbf{x}^* is a feasible solution to [P], \mathbf{y}^* is a feasible solution to [D], and

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$$

then (i) \mathbf{x}^* is an optimal solution to [P] and (ii) \mathbf{y}^* is an optimal solution to [D].



Corollary 2. If [P] is unbounded, then [D] must be infeasible.



Corollary 3. If [D] is unbounded, then [P] must be infeasible.

Proof. Similar to the previous corollary.

- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
 - See Rader p. 328 for an example
- All these theorems and corollaries apply to arbitrary primal-dual LP pairs, not just the ones we specified above

2 Strong duality

Strong Duality Theorem. Let [P] denote a primal LP and [D] its dual.

- If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.
 - If [P] and [D] both have feasible solutions, then
 - [P] has a finite optimal solution \mathbf{x}^* ;
 - [D] has a finite optimal solution \mathbf{y}^* ;
 - the optimal values of [P] and [D] are equal.
- This is an AMAZING fact
 - Useful from theoretical, algorithmic, and modeling perspectives
 - Even the simplex method implicitly uses duality: the reduced costs are essentially dual solutions that are infeasible until the last step