SA305 – Linear Programming
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Spring 2013

Lesson 30. Weak and Strong Duality

0 Warm up

Example 1. Consider the following LP and its equivalent canonical form LP:

$$\begin{array}{llcll} \text{maximize} & 2x_1 + x_2 & \quad & \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 8 & & \text{subject to} & x_1 + 2x_2 + s_1 = 8 \\ & 3x_1 + x_2 \leq 9 & & & 3x_1 + x_2 + s_2 = 9 \\ & x_1, x_2 \geq 0 & & & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

Define the decision variable vector $\mathbf{x} = (x_1, x_2, s_1, s_2)$. Solve the canonical form LP using the simplex method, with initial BFS $\mathbf{x}^0 = (3, 0, 5, 0)$ with basis $\mathcal{B}^0 = \{x_1, s_1\}$.

Example 2. Consider the following LP and its equivalent canonical form LP:

$$\begin{array}{llcll} \text{minimize} & 8y_1 + 9y_2 & \quad & \text{minimize} & 8y_1 + 9y_2 \\ \text{subject to} & y_1 + 3y_2 \geq 2 & & \text{subject to} & y_1 + 3y_2 - s_1 = 2 \\ & 2y_1 + y_2 \geq 1 & & & 2y_1 + y_2 - s_2 = 1 \\ & x_1, x_2 \geq 0 & & & y_1, y_2, s_1, s_2 \geq 0 \end{array}$$

Define the decision variable vector $\mathbf{y} = (y_1, y_2, s_1, s_2)$. Solve the canonical form LP using the simplex method, with initial BFS $\mathbf{y}^0 = (0, 1, 1, 0)$ with basis $\mathcal{B}^0 = \{y_2, s_1\}$.

1 Weak duality

- Consider the following primal-dual pair of LPs

$$
\text{[P]} \quad \begin{array}{ll} \text{maximize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}
\qquad\qquad
\text{[D]} \quad \begin{array}{ll} \text{minimize} & \mathbf{b}^\top \mathbf{y} \\ \text{subject to} & A^\top \mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0} \end{array}
$$

- Remember we constructed the dual in such a way that the multipliers $\mathbf{y}$ give us an upper bound on the optimal value of [P]

Weak Duality Theorem. Let $\mathbf{x}^*$ be a feasible solution to [P], and let $\mathbf{y}^*$ be a feasible solution to [D]. Then

$$\mathbf{c}^\top \mathbf{x}^* \le \mathbf{b}^\top \mathbf{y}^*$$

Corollary 1. If $\mathbf{x}^*$ is a feasible solution to [P], $\mathbf{y}^*$ is a feasible solution to [D], and

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$$

then (i) $\mathbf{x}^*$ is an optimal solution to [P] and (ii) $\mathbf{y}^*$ is an optimal solution to [D].

Corollary 2. If [P] is unbounded, then [D] must be infeasible.

Corollary 3. If [D] is unbounded, then [P] must be infeasible.

Proof. Similar to the previous corollary.

- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
  - See Rader p. 328 for an example
- All these theorems and corollaries apply to arbitrary primal-dual LP pairs, not just the ones we specified above

2 Strong duality

Strong Duality Theorem. Let [P] denote a primal LP and [D] its dual.

a. If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.

b. If [P] and [D] both have feasible solutions, then
   - [P] has a finite optimal solution $\mathbf{x}^*$;
   - [D] has a finite optimal solution $\mathbf{y}^*$;
   - the optimal values of [P] and [D] are equal.

- This is an AMAZING fact
- Useful from theoretical, algorithmic, and modeling perspectives
- Even the simplex method implicitly uses duality: the reduced costs are essentially dual solutions that are infeasible until the last step