Lesson 32. Maximin and Minimax Objectives

Example 1. The State of Simplex wants to divide the effort of its on-duty officers among 8 highway segments to reduce speeding incidents. You, the analyst, were able to estimate that for each highway segment $j \in \{1, \ldots, 8\}$, the weekly reduction in speeding incidents is $r_j + s_j x$, where x is the number of officers assigned to segment j. Due to local ordinances, there is an upper bound u_j on the number of officers assigned to highway segment j per week, for $j \in \{1, \ldots, 8\}$. There are 25 officers per week to allocate.

The State of Simplex has decided that it wants to maximize the worst-case reduction in speeding incidents among all highway segments. Write a linear program that allocates officers to highway segments according to this objective.

1 Maximin objective functions

- Maximin objective function:

$$\text{maximize} \quad \min\left\{ \sum_{j=1}^{n} a_{ij}x_j + b_i : i = 1, \ldots, m \right\}$$

  where $x_1, \ldots, x_n$ are decision variables, and a_{ij} and b_i are constants for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$

  - Add auxiliary decision variable z
  - Change objective:

$$\text{maximize} \quad z$$

  - Add constraints:

$$z \leq \sum_{j=1}^{n} a_{ij}x_j + b_i \qquad \text{for } i \in \{1, \ldots, m\}$$

2 Minimax objective functions

- Minimax objective function:

$$\text{minimize} \quad \max\left\{ \sum_{j=1}^{n} a_{ij}x_j + b_i : i = 1, \ldots, m \right\}$$

  where $x_1, \ldots, x_n$ are decision variables, and a_{ij} and b_i are constants for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$

  - Add auxiliary decision variable z
  - Change objective:

$$\text{minimize} \quad z$$

  - Add constraints:

$$z \geq \sum_{j=1}^{n} a_{ij}x_j + b_i \qquad \text{for } i \in \{1, \ldots, m\}$$

Example 2. Captain Hook and Captain Sparrow have "found" some treasure. They can't agree on how to split their newly found riches, and so they have asked you for help. To start, you ask them to assign "value points" to all of the items under consideration, so that the points add up to 100. Their valuations are shown below:

	Item	Captain Hook	Captain Sparrow
1	Gold coins	40	30
2	Jewels	10	20
3	Tobacco	10	20
4	Spices	40	30

Assume that the items can be divided fractionally. For example, Captain Hook can receive 60% of the gold coins, while Captain Sparrow can receive the remaining 40%. In this case, Captain Hook gets $0.6 \times 40 = 24$ value points, and Captain Sparrow gets $0.4 \times 30 = 12$ value points.

Formulate a linear program to determine how to allocate the treasure between Captain Hook and Captain Sparrow, in a way that maximizes the minimum total value points of any captain.