Lesson 34. An Economic Interpretation of LP Duality

Today

- More intuition behind the dual: economic interpretation
- Complementary slackness

Warm up

The Fulkerson Furniture Company produces desks, tables, and chairs. Each type of furniture requires a certain amount of lumber, finishing, and carpentry:

Resource	Desk	Table	Chair	Available
Lumber (sq ft)	8	6	2	48
Finishing (hrs)	3	2	1	20
Carpentry (hrs)	2	2	1	8
Profit ($)	60	30	20	

Assume that all furniture produced is sold, and that fractional solutions are acceptable. Write a linear program to determine how much furniture Fulkerson should produce in order to maximize its profits.

- Decision variables:

- Fulkerson's LP:

Economic interpretation of the dual LP

- Suppose an entrepreneur wants to purchase all of Fulkerson's resources (lumber, finishing, carpentry)
- What prices should she offer for the resources that will entice Fulkerson to sell?

- Define decision variables:

y_1 = price of 1 sq. ft. lumber
y_2 = price of 1 hour of finishing
y_3 = price of 1 hour of carpentry

- To buy all of Fulkerson's resources, entrepreneur pays:

- Entrepreneur wants to minimize costs
- Entrepreneur needs to offer resource prices that will entice Fulkerson to sell
- One desk uses
  - 8 sq. ft. of lumber
  - 3 hours of finishing
  - 2 hours of carpentry
  - One desk has profit of $60

⇒ Entrepreneur should pay at least $60 for this combination of resources:

- One table uses
  - 6 sq. ft. of lumber
  - 2 hours of finishing
  - 2 hours of carpentry
  - One table has profit of $30

⇒ Entrepreneur should pay at least $30 for this combination of resources:

- One chair uses
  - 2 sq. ft. of lumber
  - 1 hours of finishing
  - 1 hours of carpentry
  - One chair has profit of $20

⇒ Entrepreneur should pay at least $20 for this combination of resources:

- Increasing the availability of the resources potentially increases the maximum profits Fulkerson can achieve

⇒ Entrepreneur should pay nonnegative amounts for each resource:

- Putting this all together, we get:

$$\begin{array}{lrll} \min & 48y_1 + 20y_2 + 8y_3 & & \\ \text{s.t.} & 8y_1 + 3y_2 + 2y_3 \geq 60 & & (x_1\text{: desks}) \\ & 6y_1 + 2y_2 + 2y_3 \geq 30 & & (x_2\text{: tables}) \\ & 2y_1 + y_2 + y_3 \geq 20 & & (x_3\text{: chairs}) \\ & y_1, \quad y_2, \quad y_3 \geq 0 & & \end{array}$$

- This is the dual of Fulkerson's LP!
- In summary:
  - Dual variables $\Leftrightarrow$ "fair" prices for 1 unit of the associated resource
  - Known as **marginal prices** or **shadow prices**
- Strong duality
  - maximum revenue from selling furniture = minimum cost of purchasing resources

  ⇒ equilibrium between market for furniture and market for resources
- This kind of economic interpretation is trickier for LPs with different types of constraints and variable bounds

Complementary slackness

- Optimal solution to Fulkerson's LP: $x_1 = 4,\ x_2 = 0,\ x_3 = 0$
- Resources used:

  lumber: $32 < 48$ finishing: $12 < 20$ carpentry: $8 = 8$
- How much would you pay for an extra sq. ft. of lumber?
- How much would you pay for an extra hour of finishing?

- Resource not fully utilized in optimal solution

  $\Rightarrow$ marginal price = 0

- **Primal complementary slackness:** either
  - the primal optimal solution makes a main inequality constraint active, or
  - the corresponding dual variable has optimal value = 0
- Same logic applies to the dual
- Dual constraints $\Leftrightarrow$ Primal decision variables
- **Dual complementary slackness:** either
  - primal decision variable has optimal value = 0, or
  - the corresponding dual constraint is active in a dual optimal solution