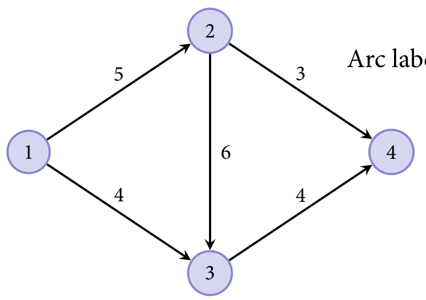
Lesson 36. The Shortest Path Problem, cont.

1 Last time ...

- What is the shortest path from vertex 1 to vertex 4?



  - $V = \{1, 2, 3, 4\}$
  - $A = \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$

- Integer program:
  - Decision variables: $x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in the selected path} \\ 0 & \text{otherwise} \end{cases}$ for $(i,j) \in A$

minimize	$5x_{12} + 4x_{13} + 6x_{23} + 3x_{24} + 4x_{34}$	(total length of selected arcs)
subject to	$x_{12} + x_{13} = 1$	(exactly 1 arc out of vertex 1)
	$x_{24} + x_{34} = 1$	(exactly 1 arc into vertex 4)
	$x_{12} = x_{23} + x_{24}$	(what goes into vertex 2 = what goes out of vertex 2)
	$x_{13} + x_{23} = x_{34}$	(what goes into vertex 3 = what goes out of vertex 3)
	$x_{ij} \in \{0,1\}$ for $(i,j) \in A$	

2 An integer program for the shortest path problem

- Let's generalize what we did last time to any shortest path problem
- Network (V, A)
- Length c_{ij} for each arc $(i, j) \in A$
- Source vertex s, sink vertex t
- Decision variables:
- Integer program:
- This integer program holds for any shortest path problem
  - You just need to specify values for the symbolic input parameters (V, A), s, t, and c_{ij} for $(i, j) \in A$
- We can solve the shortest path integer program above by replacing the binary constraints with nonnegativity bounds and solving the resulting linear program, that is:
- Note that this is a special property of the shortest path integer program
  - **In general, we cannot solve integer programs by replacing the integrality constraints with continuous constraints**

3 Modeling optimization problems as shortest path problems

- Some problems that have nothing to do with networks on the surface can be modeled as shortest path problems

Example 1. The Fulkerson night bus begins running at 7pm and continues until 2am. Several drivers can be used, but only one should be on duty at any time. If a shift starts at or before 9pm, a regular driver can be obtained for a 4-hour shift for a total cost of \$50. Otherwise, part-time drivers can be obtained for a 3-hour shift at a cost of \$40, or 2-hour shifts at a cost of \$30. Formulate the problem of computing a minimum total cost nightly driver shift schedule as a shortest path problem.

4 Food for thought

- Consider the linear program for the shortest path problem. What does its dual look like?