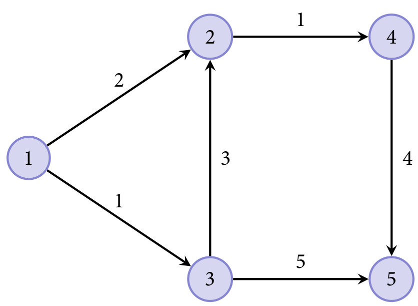
Lesson 37. The Shortest Path Interdiction Problem

1 Warm up

Example 1. Write a linear program that finds the shortest path from vertex 1 to vertex 5 in the network below.



Arc labels are lengths

Example 2. Write the dual of the LP you wrote in Example 1.

2 Optimality conditions for the shortest path problem

- Input parameters:
  - Network (V, A)
  - Arc lengths c_{ij} for $(i, j) \in A$
  - Source vertex s, sink vertex t
- LP for shortest path problem:

$$
\begin{array}{lrll}
\text{minimize} & & \displaystyle\sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{subject to} & & \displaystyle - \sum_{(s,j)\in A} x_{sj} = -1 & \\
& \displaystyle\sum_{(k,i)\in A} x_{ki} - & \displaystyle\sum_{(i,j)\in A} x_{ij} = 0 & \text{for } i \in V \setminus \{s, t\} \\
& & \displaystyle\sum_{(i,t)\in A} x_{it} = 1 & \\
& & x_{ij} \geq 0 & \text{for } (i, j) \in A
\end{array}
$$

- The dual of this LP is:

- These dual constraints are used to design very fast algorithms for the shortest path problem

3 Shortest path interdiction

- Suppose that you are defending the network given in Example 1
- Your enemy wants to move from vertex 1 to vertex 5 in the shortest way possible
- You can obstruct the enemy by increasing the arc lengths
- You have a budget: the total increase in arc lengths must be at most 10
- You want to determine how to increase the arc lengths so that the length of the shortest path in the resulting network is maximized
- We can write an optimization model using the LP from Example 1 as a starting point
- Additional decision variables:
- Optimization model:
- How can we deal with the hierarchical structure (i.e. max min) of this model?
- Take the dual of the "inner" optimization model:
- LP duality can help us model complex problems as linear programs

3.1 A general description of the shortest path interdiction problem

- Input parameters:
  - Network (V, A)
  - Arc lengths c_{ij} for $(i, j) \in A$
  - Source vertex s, sink vertex t
  - Per-unit interdiction cost d_{ij} for $(i, j) \in A$
  - Interdiction budget B
- Hierarchical optimization model:

- Equivalent linear program using LP duality: