Lesson 38. Review

Problem 1. Consider the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 10x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 + 4x_3 + 21x_4 = 13 \\ & 2x_1 + 6x_2 - 2x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Let the current basis be $\{x_1, x_3\}$.

a. Compute the current BFS.
b. Compute all the simplex directions at the current BFS.
c. Determine whether each of the simplex directions is improving.
d. Choose the "most improving" simplex direction and determine the maximum step size that preserves feasibility in that direction, and the new basis that would result after such a step.

Problem 2. Consider the following LP:

$$\begin{aligned} [\text{P}] \quad \max \quad & 3x_1 + 4x_2 + x_3 + 5x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 + 2x_4 \leq 5 \\ & 2x_1 + 3x_2 + x_3 + 3x_4 \leq 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

a. Write the dual of [P].
b. Graphically determine an optimal solution of the dual of [P].
c. Use complementary slackness to determine which two decision variables of [P] are equal to 0 in an optimal solution. Use this information to write an LP that determines the optimal values of the remaining two decision variables of [P].
d. Graphically determine an optimal solution for the LP in part c. Write an optimal solution for [P].

Problem 3. Gomoryco processes crude oil into aviation fuel and heating oil. It costs c dollars to purchase 1000 barrels of crude oil, which is then distilled and yields 750 barrels of aviation fuel and 250 barrels of heating oil. Output from the distillation may be sold directly or processed in the catalytic cracker. If sold after distillation without further processing, aviation fuel sells for p_a dollars per 1000 barrels, and heating oil sells for p_h dollars per 1000 barrels. It takes t_a hours to process 1000 barrels of aviation fuel in the catalytic cracker, and these 1000 barrels can be sold for q_a. It takes t_h minutes to process 1000 barrels of heating oil in the cracker, and these 1000 barrels can be sold for q_h. Each day, at most B barrels of crude oil can be purchased, and T hours of cracker time are available. Formulate a linear program that maximizes Gomoryco's profits.

Problem 4. Dijkstra Pharmaceuticals must supply 30 batches of its new medication in the next quarter, then 25, 10, and 35 in successive quarters. Each quarter in which the company makes product requires a $100K setup cost, plus $3K per batch produced. Batches can be held in inventory at a cost of $5K per batch per quarter. Dijkstra seeks a minimum total cost production plan. Explain why Dijkstra's problem can be formulated as a shortest path problem.

Hint. Create a set of vertices $\{1, \ldots, 5\}$. For $k \in \{1, \ldots, 5\}$, vertex k represents reaching quarter k with all earlier demand fulfilled and no inventory on hand.

Problem 5. Suppose you are applying the simplex method to a canonical form LP with objective

$$\text{minimize} \quad 3w_1 + 11w_2 - 8w_3$$

Determine whether each of the following simplex directions for w_4 leads to a conclusion that the given LP is unbounded. Why?

a. $\mathbf{d}^{w_4} = (1, 0, -4, 1)$
b. $\mathbf{d}^{w_4} = (1, 3, 0, 1)$
c. $\mathbf{d}^{w_4} = (1, 0, 3, 1)$
d. $\mathbf{d}^{w_4} = (-1, 1, -2, 1)$

Problem 6. Santa Claus has a set of presents P that he wants to distribute to a set of children C. Let v_{ij} be the happiness value that child i has for present j, for all $i \in C$ and $j \in P$. In addition, let b_j be the number of present j that Santa has available. Santa's objective is to distribute presents in a way that maximizes the happiness of the least lucky child. Formulate a linear program that helps Santa meet his objective. Assume that presents can be distributed fractionally.

Note. The "Santa Claus problem" is actually studied in the operations research and computer science literature as a fundamental resource allocation problem. It has particular importance in scheduling applications. See `http://dx.doi.org/10.1145/1132516.1132522` for an example.