P1.

a. Current basis $= \{x_1, x_3\} \Rightarrow x_2 = x_4 = 0$ at current BFS:

$$\left.\begin{aligned} -x_1 + 4x_3 &= 13 \\ 2x_1 &= 2 \end{aligned}\right\} \Rightarrow x_1 = 1, \quad x_3 = \frac{14}{4} = \frac{7}{2}$$

$\Rightarrow$ current BFS $= (1, 0, \frac{7}{2}, 0)$.

b. $\vec{d}^{x_2} = (d_{x_1}, 1, d_{x_3}, 0)$

$$-d_{x_1} + 4d_{x_3} = -1$$
$$2d_{x_1} = -6$$

$\Rightarrow d_{x_1} = -3, \; d_{x_3} = -1$

$\Rightarrow \vec{d}^{x_2} = (-3, 1, -1, 0)$

$\vec{d}^{x_4} = (d_{x_1}, 0, d_{x_3}, 1)$

$$-d_{x_1} + 4d_{x_3} = -21$$
$$2d_{x_1} = 2$$

$\Rightarrow d_{x_1} = 1, \; d_{x_3} = -5$

$\Rightarrow \vec{d}^{x_4} = (1, 0, -5, 1)$.

c. $\bar{c}_{x_2} = -30 + 1 = -29$

$\Rightarrow \vec{d}^{x_2}$ not improving

$\bar{c}_{x_4} = 10$

$\Rightarrow \vec{d}^{x_4}$ improving.

d. Choose $\vec{d}^{x_4}$: x_4 is the entering variable

$\lambda_{max} = \min\left\{\frac{7/2}{5}\right\} = \frac{7}{10}$ $\qquad x_3$ is the leaving variable.

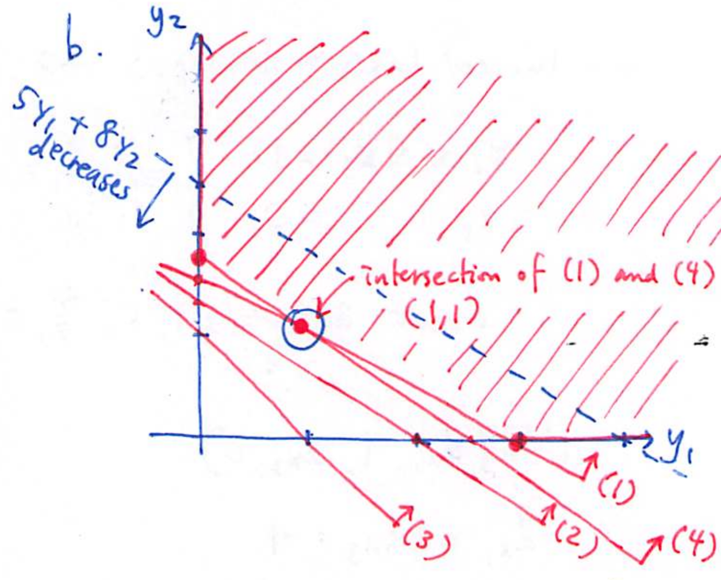
$\Rightarrow$ new BFS $= (1, 0, \frac{7}{2}, 0) + \frac{7}{10}(1, 0, -5, 1) = (\frac{17}{10}, 0, 0, \frac{7}{10})$

new basis $= \{x_1, x_4\}$.

P2. a. $\min \quad 5y_1 + 8y_2$

s.t.
$$\begin{aligned} y_1 + 2y_2 &\geq 3 \quad (1) \\ 2y_1 + 3y_2 &\geq 4 \quad (2) \\ y_1 + y_2 &\geq 1 \quad (3) \\ 2y_1 + 3y_2 &\geq 5 \quad (4) \\ y_1, y_2 &\geq 0. \end{aligned}$$

b.



optimal soln: $y_1^* = 1$, $y_2^* = 1$.

c. Dual complementary slackness: dual constraint not active
$\Rightarrow$ corresponding primal variable = 0.

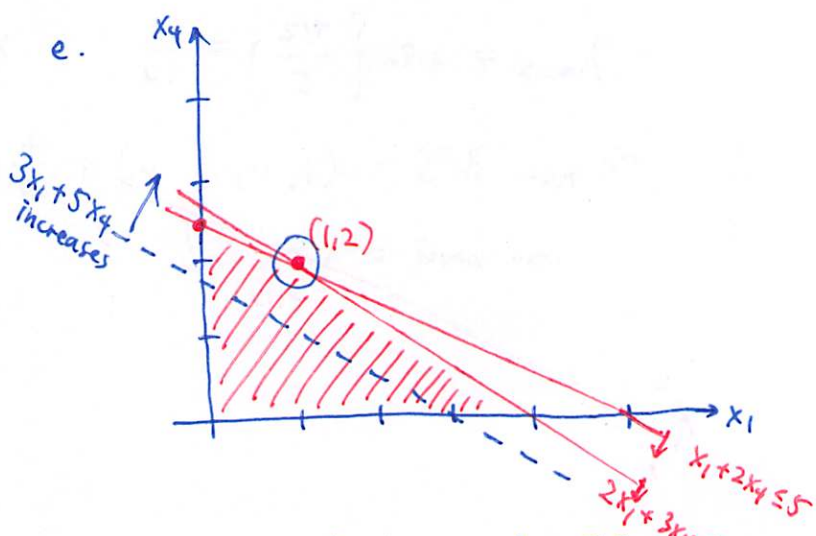
From part b, it is clear that constraints (2) and (3) are not active at (1,1). $\Rightarrow$ In an optimal solution to [P], we must have $x_2 = x_3 = 0$.

d. modified [P]:

$\max \quad 3x_1 + 5x_4$

s.t.
$$\begin{aligned} x_1 + 2x_4 &\leq 5 \\ 2x_1 + 3x_4 &\leq 8 \\ x_1, x_4 &\geq 0 \end{aligned}$$

e.



optimal soln to modified [P]:
$x_1^* = 1$, $x_4^* = 2$

optimal soln. to [P]: $(1, 0, 0, 2)$

P3. Symbolic input parameters:

c = cost of crude oil per 1000 barrels

p_a = price of unprocessed aviation fuel per 1000 barrels

p_h = price of unprocessed heating oil per 1000 barrels

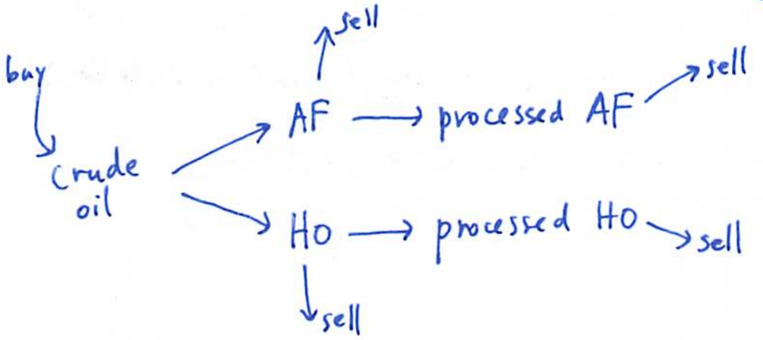
q_a = price of processed aviation fuel per 1000 barrels

q_h = price of processed heating oil per 1000 barrels

t_a = time to process aviation fuel per 1000 barrels

t_h = time to process heating oil per 1000 barrels

B = available crude oil, in 1000s.



Decision variables: (in 1000 barrels)

z = amt. of crude oil to buy

x_a = amt. of unprocessed AF to sell

x_h = amt. of unprocessed HO to sell

y_a = amt. of processed AF to sell

y_h = amt. of processed HO to sell.

Model:

$$\max \quad p_a x_a + p_h x_h + q_a y_a + q_h y_h - cz \quad \text{(total profit)}$$

$$\text{s.t.} \quad \frac{3}{4} z = x_a + y_a \quad \text{(crude oil} \to \text{AF)}$$

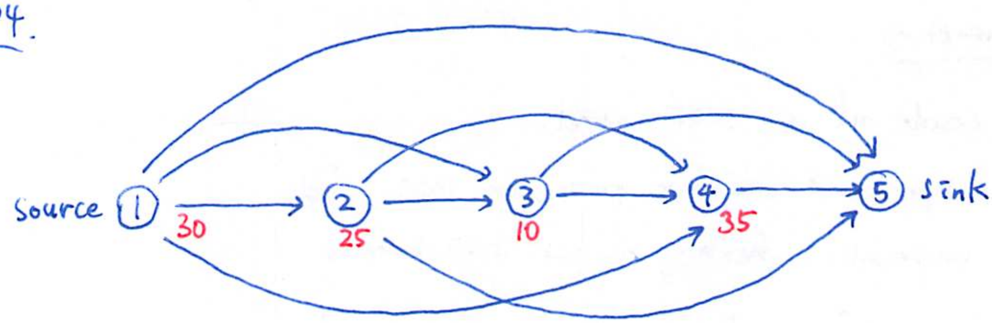
$$\frac{1}{4} z = x_h + y_h \quad \text{(crude oil} \to \text{HO)}$$

$$t_a y_a + t_h y_h \le T \quad \text{(cracker time)}$$

$$z \le B \quad \text{(available crude oil)}$$

$$x_a, y_a, x_h, y_h, z \geq 0.$$

P4.



vertices $\longleftrightarrow$ quarters.

numbers in red next to vertices = demand in that quarter

Arc (i,j) represents: "produce in quarter i to cover demand in quarters $i, \ldots, j-1$."

For example, arc $(1,3)$: produce $30 + 25$ in quarter 1

30 units used immediately to satisfy demand in quarter 1

25 units held over to satisfy demand in quarter 2.

total cost: $100 + 3(30+25) + 5(25) = 390$.

arc	costs
$(1,2)$	$100 + 3(30)$
$(1,3)$	$100 + 3(30+25) + 5(25)$
$(1,4)$	$100 + 3(30+25+10) + 5(25) + (5+5)(10)$
$(1,5)$	$100 + 3(30+25+10+35) + 5(25) + (5+5)(10) + (5+5+5)(10)$.
$(2,3)$	$100 + 3(25)$
$(2,4)$	$100 + 3(25+10) + 5(10)$
$(2,5)$	$100 + 3(25+10+35) + 5(10) + (5+5)(35)$
$(3,4)$	$100 + 3(10)$
$(3,5)$	$100 + 3(10+35) + 5(35)$
$(4,5)$	$100 + 3(35)$

demand in quarter 3 has to be held for $\underline{\underline{2}}$ quarters.

A shortest path from vertex 1 to vertex 5 in the above network w/ the above arc costs corresponds to a minimum total cost production plan.

P5. The objective fn. vector $\vec{c} = (3, 11, -8, 0)$

a. $\vec{d}^{W4} = (1, 0, -4, 1)$ does NOT lead to a conclusion that the LP is unbounded, since its components are not all nonnegative.

b. $\vec{d}^{W4} = (1, 3, 0, 1)$ has associated reduced cost $\bar{c}_{W4} = 36$. Since the LP is minimizing, $\vec{d}^{W4}$ is not improving. So, even though all components of $\vec{d}^{W4}$ are nonnegative, we cannot conclude that the LP is unbounded.

c. $\vec{d}^{W4} = (1, 0, 3, 1)$ has associated reduced cost $\bar{c}_{W4} = -21$, and so $\vec{d}^{W4}$ is improving, with all nonnegative components. Therefore, we can conclude that the LP is unbounded.

d. $\vec{d}^{W4} = (-1, 1, -2, 1)$. Similar to part a.

P6. Symbolic input parameters:

P = set of presents
C = set of children
v_{ij} = happiness of child i w/present j for $i \in C$, $j \in P$
b_j = # present j available for $j \in P$.

Decision variables: x_{ij} = # present j given to child i for $i \in C$, $j \in P$.

Model:

$$\max \quad \min \Big\{ \underbrace{\sum_{j \in P} v_{ij} x_{ij}}_{\text{happiness of child } i} : i \in C \Big\}$$

$$\text{s.t.} \quad \sum_{i \in C} x_{ij} \le b_j \quad \text{for } j \in P \quad \text{(available presents)}$$

$$x_{ij} \ge 0 \quad \text{for } i \in C, j \in P.$$

convert to LP

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z \le \sum_{j \in P} v_{ij} x_{ij} && \text{for } i \in C \\ & \sum_{i \in C} x_{ij} \le b_j && \text{for } j \in P \\ & x_{ij} \ge 0 && \text{for } i \in C, j \in P. \end{aligned}$$