SA305 – Linear Programming — Spring 2014
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Lesson 18. Improving Search: Convexity and Optimality

0 Warm up – last time...

- General optimization model with only continuous variables
  - Decision variables: $\mathbf{x} = (x_1, \dots, x_n)$
  - Multivariable functions in $\mathbf{x}$: $f(\mathbf{x})$, $g_i(\mathbf{x})$ for $i \in \{1, \dots, m\}$, not necessarily linear
  - Constant scalars: b_i for $i \in \{1, \dots, m\}$

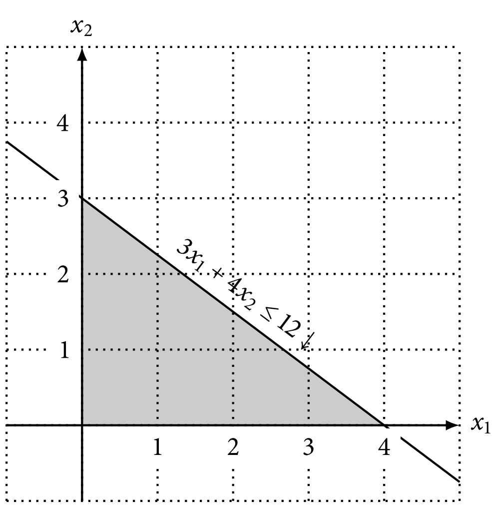
$$\begin{aligned} \text{minimize / maximize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \begin{Bmatrix} \le \\ \ge \\ = \end{Bmatrix} b_i \quad \text{for } i \in \{1, \dots, m\} \end{aligned} \qquad (*)$$

- Moving from one solution to the next: $\mathbf{x}^{k+1} = \mathbf{x}^k + \lambda \mathbf{d}$
- $\mathbf{d}$ is **improving** at $\mathbf{x}^k$ "if it points towards solutions with better objective function value"
- $\mathbf{d}$ is **feasible** at $\mathbf{x}^k$ "if it points towards feasible solutions"

Example 1. Consider the LP below and the graph of its feasible region. Let $\mathbf{x}^k = (0, 2)$ and $\mathbf{d} = (0, -1)$.

a. Is $\mathbf{d}$ a feasible direction at $\mathbf{x}^k$? Why?
b. Let $\lambda = 1$. Compute $\mathbf{x}^{k+1}$.
c. What is the change in value from $\mathbf{x}^k$ to $\mathbf{x}^{k+1}$?
d. Is $\mathbf{d}$ an improving direction at $\mathbf{x}^k$? Why?

$$\begin{aligned} \text{minimize} \quad & 3x_1 + x_2 \\ \text{subject to} \quad & 3x_1 + 4x_2 \le 12 && (1) \\ & x_1 \ge 0 && (2) \\ & x_2 \ge 0 && (3) \end{aligned}$$



1 Today...

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1 Find an initial feasible solution x^0
2 Set k = 0
3 while x^k is not locally optimal do
4     Determine a new feasible solution x^{k+1} that improves the objective value at x^k
5     Set k = k + 1
6 end while
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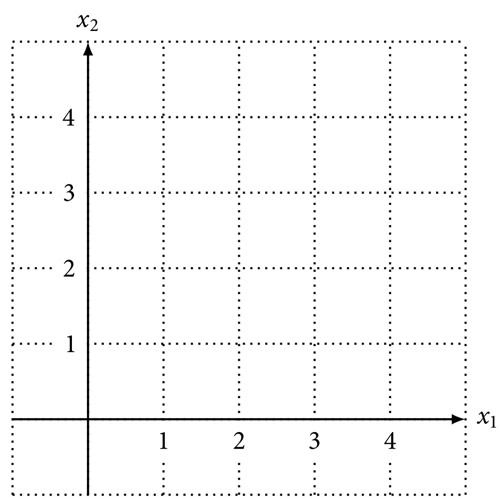
- Step 3 – Improving search converges to local optimal solutions, which aren't necessarily globally optimal
- Wishful thinking: when are all local optimal solutions are in fact globally optimal?

2 Convex sets

Example 2. Let $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (4, 3)$. Plot $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ for $\lambda \in \{0, 1/3, 2/3, 1\}$.



- Given two solutions $\mathbf{x}$ and $\mathbf{y}$, the **line segment** joining them is

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \quad \text{for } \lambda \in [0, 1]$$

- A feasible region S is **convex** if for all $\mathbf{x}, \mathbf{y} \in S$, then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$ for all $\lambda \in [0, 1]$
  - A feasible region is convex if for any two solutions in the region, all solutions on the line segment joining these solutions are also in the region
- Geometrically: convex vs. nonconvex

Example 3. Show that the feasible region of the LP in Example 1 is convex.

Proof.

- Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be arbitrary points in the feasible region
- In other words, $\mathbf{x}$ and $\mathbf{y}$ satisfy (1), (2), (3)
- We need to show $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ also satisfies (1), (2), (3) for any $\lambda \in [0,1]$
- Note that

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \boxed{}$$

- One constraint at a time: does $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ satisfy (1)?

$$\begin{aligned} 3\big(\lambda x_1 + (1-\lambda)y_1\big) + 4\big(\lambda x_2 + (1-\lambda)y_2\big) &= \lambda\big(3x_1 + 4x_2\big) + (1-\lambda)\big(3y_1 + 4y_2\big) \\ \\ &\leq 12\lambda + 12(1-\lambda) \\ &= 12 \end{aligned}$$

- We can show $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ also satisfies (2) and (3) in a similar fashion □

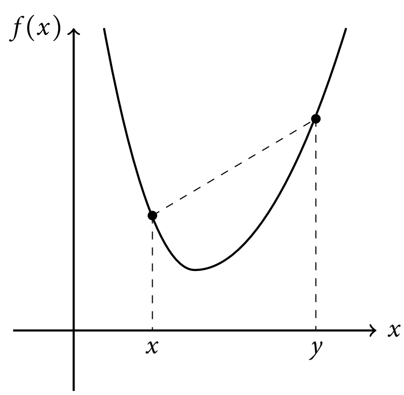
- **In general, the feasible region of an LP is convex**

3 Convex functions

- Given a convex feasible region S, a function $f(\mathbf{x})$ is **convex** if for all solutions $\mathbf{x}, \mathbf{y} \in S$ and for all $\lambda \in [0,1]$

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y})$$

- Example:



Example 4. Show that the objective function of the LP in Example 1 is convex.

Proof.

- Let $f(\mathbf{x}) = 3x_1 + x_2$
- For any $\mathbf{x}$ and $\mathbf{y}$, we have:

$$\begin{aligned} f\big(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\big) &= 3\big(\lambda x_1 + (1-\lambda)y_1\big) + \big(\lambda x_2 + (1-\lambda)y_2\big) \\ &= \lambda\big(3x_1 + x_2\big) + (1-\lambda)\big(3y_1 + y_2\big) \\ \\ &= \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \qquad \square \end{aligned}$$

- **In general, the objective function of an LP – a linear function – is convex**

4 Minimizing convex functions over convex sets

Big Theorem. Consider the minimization version of the optimization model (∗). Suppose f is convex and the feasible region is convex. If an improving search algorithm stops at a local minimum $\mathbf{x}$, then $\mathbf{x}$ is a global minimum.

Proof.

- By contradiction – suppose $\mathbf{x}$ is not a global minimum
- Then there must be another feasible solution $\mathbf{y} \in S$ such that $f(\mathbf{y}) < f(\mathbf{x})$
- Take $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ really close to $\mathbf{x}$ (λ really close to 1)
- Since the feasible region S is convex, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also in S (and therefore feasible)
- We have that:

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &\leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) && \text{(since } f \text{ is convex)} \\ &< \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{x}) && \text{(since } f(\mathbf{y}) < f(\mathbf{x})\text{)} \\ &= f(\mathbf{x}) \end{aligned}$$

- Therefore: $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < f(\mathbf{x})$
- $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is a feasible solution in the neighborhood of $\mathbf{x}$ with better objective value than $\mathbf{x}$
- This contradicts $\mathbf{x}$ being a local minimum! $\mathbf{x}$ must be a global minimum. $\square$

- Since the objective function of an LP is convex, and the feasible region of an LP is convex:

Big Corollary 1. A global optimal solution of a minimizing linear program can be found with an improving search algorithm.

- A similar theorem and corollary exists when maximizing concave functions over convex sets
  - See pages 222–225 in Rader for details

Big Corollary 2. A global optimal solution of a maximizing linear program can be found with an improving search algorithm.