SA305 – Linear Programming
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Spring 2014

Lesson 21. Geometry and Algebra of "Corner Points", cont.

1 Warm up and overview

- Last time: "corner points" of the feasible region of an LP
- **Polyhedron**: set of vectors $\mathbf{x}$ that satisfy a finite collection of linear constraints
    - Feasible region of an LP $\Leftrightarrow$ polyhedron
- **Extreme point $\mathbf{x}$** of polyhedron S: any line segment through $\mathbf{x}$ has an endpoint outside of S
- **Linearly independent constraints** of a polyhedron: rows of constraint LHS coefficient matrix are LI
- **Basic solution $\mathbf{x}$** of polyhedron S with n decision variables:
  (a) $\mathbf{x}$ satisfies all equality constraints
  (b) at least n constraints are active at $\mathbf{x}$ and are linearly independent
- **Basic feasible solution (BFS) $\mathbf{x}$** of polyhedron S: $\mathbf{x}$ is a basic solution and satisfies all constraints of S
- "Corner points" = extreme points = basic feasible solutions

Example 1. Consider the polyhedron S defined below.

$$S = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{rr} x_1 + 3x_2 \leq 4 & (1) \\ x_1 \geq 0 & (2) \\ x_2 \geq 0 & (3) \end{array} \right\}$$

a. Verify that constraints (1) and (3) are linearly independent.
b. Compute the basic solution $\mathbf{x}$ active at constraints (1) and (3). Is $\mathbf{x}$ a BFS? Why?
c. In words, how would you find all the basic feasible solutions of S?

- Today: When are extreme points adjacent? Is there always an optimal solution to an LP that is an extreme point?

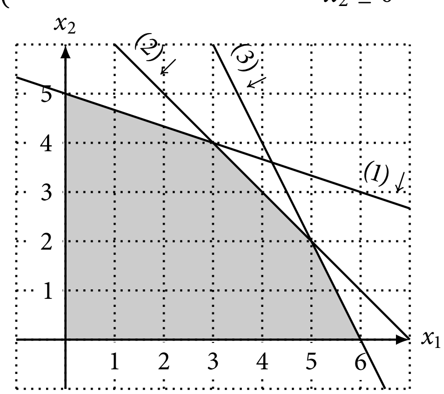
2 Adjacency

- An **edge** of a polyhedron S with n decision variables is the set of solutions in S that are active at $(n-1)$ linearly independent constraints

Example 2. Consider the polyhedron and its graph below.

a. How many linearly independent constraints need to be active for an edge of this polyhedron?
b. Describe the edge associated with constraint (2).

$$S = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{rr} x_1 + 3x_2 \leq 15 & (1) \\ x_1 + x_2 \leq 7 & (2) \\ 2x_1 + x_2 \leq 12 & (3) \\ x_1 \geq 0 & (4) \\ x_2 \geq 0 & (5) \end{array} \right\}$$



- Two extreme points of a polyhedron S with n decision variables are **adjacent** there are $(n-1)$ common linearly independent constraints at active both extreme points
  - Equivalently, two extreme points are adjacent if the line segment joining them is an edge of S

Example 3. Consider the polyhedron in Example 2.

a. Verify that $(3,4)$ and $(5,2)$ are adjacent extreme points.
b. Verify that $(0,5)$ and $(6,0)$ are not adjacent extreme points.

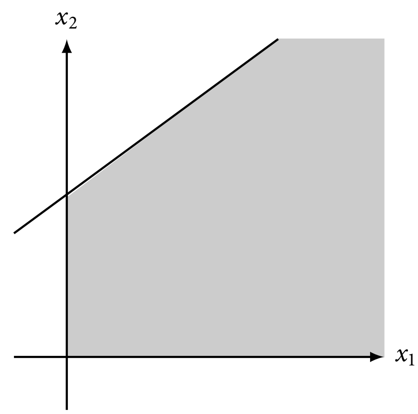
- We can move between adjacent extreme points by "swapping" active linearly independent constraints

3 Extreme points are good enough: the fundamental theorem of linear programming

Big Theorem. Let S be a polyhedron with at least 1 extreme point. Consider the LP that maximizes a linear function $\mathbf{c}^\top \mathbf{x}$ over $\mathbf{x} \in S$. Then this LP is unbounded, or attains its optimal value at some extreme point of S.

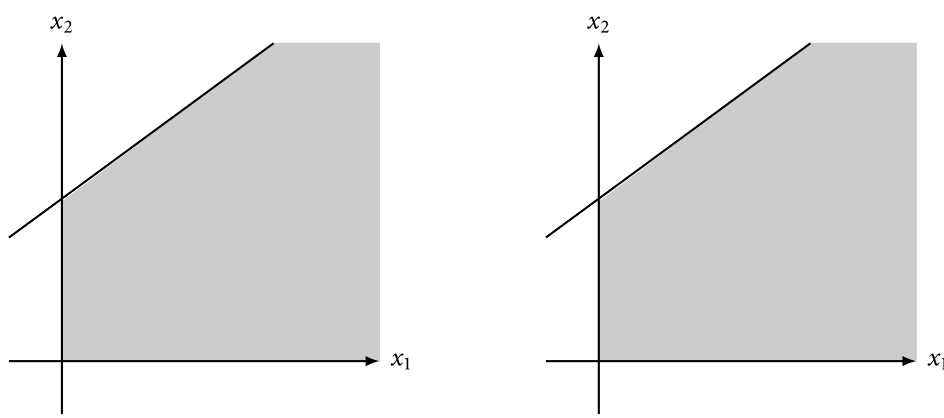
"Proof" by picture.

- Assume the LP has finite optimal value
- The optimal value must be attained at the boundary of the polyhedron, otherwise:



⇒ The optimal value is attained at an extreme point or "in the middle of a boundary"

- If the optimal value is attained "in the middle of a boundary", there must be multiple optimal solutions, including an extreme point



⇒ The optimal value is always attained at an extreme point □

- **For LPs, we only need to consider extreme points as potential optimal solutions**
- It is still possible for an optimal solution to an LP to not be an extreme point
- If this is the case, there must be another optimal solution that is an extreme point

4 Food for thought

- Last time, we saw that a polyhedron may have no extreme points
- We need to be a little careful with these conclusions – what if the Big Theorem doesn't apply?
- Next time: we will learn how to convert any LP into an equivalent LP that has at least 1 extreme point, so we don't have to be (so) careful