Lesson 23. Basic Solutions in Canonical Form LPs

1 Overview

Recall that an LP in **canonical form** looks like this:

$$\begin{aligned} \text{minimize / maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{CF}$$

Note that

(a) all the general constraints are equalities, and
(b) all the decision variables are constrained to be nonnegative.

Also, recall that a solution $\mathbf{x}$ of an LP with n decision variables is a **basic solution** if

(a) it satisfies all equality constraints, and
(b) at least n constraints are active at $\mathbf{x}$ and are linearly independent.

The solution $\mathbf{x}$ is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of the LP.

Today, we will investigate what basic solutions look like for canonical form LPs.

2 Example

Consider the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 3x + 8y \\ \text{subject to} \quad & x + 4y + s_1 = 20 & (1) \\ & x + y + s_2 = 9 & (2) \\ & 2x + 3y + s_3 = 20 & (3) \\ & x \geq 0 & (4) \\ & y \geq 0 & (5) \\ & s_1 \geq 0 & (6) \\ & s_2 \geq 0 & (7) \\ & s_3 \geq 0 & (8) \end{aligned}$$

Identify the matrix A and the vectors $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{b}$ in the above canonical form LP. *Hint.* Let $\mathbf{x} = (x, y, s_1, s_2, s_3)$.

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 3 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 20 \\ 9 \\ 20 \end{bmatrix}$$

Suppose $\mathbf{x}$ is a basic solution. How many linearly independent constraints must be active at $\mathbf{x}$? How many of these must be equality constraints, and how many of these must be nonnegativity bounds?

Since there are $n = 5$ decision variables, there must be at least 5 LI constraints active at $\vec{x}$. 3 of these must be equality constraints (since a basic solution must satisfy all equality constraints). Therefore, at least 2 of the active LI constraints must be nonnegativity bounds.

Verify that the constraints (1), (2), (3), (6), and (8) are linearly independent. (Use a calculator or MATLAB.)

LHS constraint matrix of (1), (2), (3), (6), (8):

$$L = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\det(L) = -5 \Rightarrow$ (1), (2), (3), (6), (8) are LI

Let's compute the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8). Since the basic solution is active at the nonnegativity bounds (6) and (8), s_1 and s_3 are "forced" to be zero. The other variables, x, y, and s_2 are potentially nonzero.

Substituting $s_1 = 0$ and $s_3 = 0$ into the other constraints (1), (2), and (3), we get:

$$\begin{aligned} x + 4y + (0) \qquad\qquad &= 20 \\ x + \ \ y \qquad + s_2 \qquad &= 9 \\ 2x + 3y \qquad\qquad + (0) &= 20 \end{aligned} \tag{$*$}$$

Let B be the submatrix of A consisting of columns corresponding to variables x, y, and s_2:

$$B = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

Verify that the columns of B linearly independent. (Use a calculator or MATLAB.)

$\det(B) = 5 \Rightarrow$ columns of B are LI

Note that $(*)$ can be written as

$$B\begin{pmatrix} x \\ y \\ s_2 \end{pmatrix} = \mathbf{b} \tag{$**$}$$

Use B^{-1} (it exists, right?) to find the values of x, y, and s_2. (Use a calculator or MATLAB.)

$$B^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0.4 & 0 & -0.2 \\ 0.2 & 1 & -0.6 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ s_2 \end{bmatrix} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0.4 & 0 & -0.2 \\ 0.2 & 1 & -0.6 \end{bmatrix} \begin{bmatrix} 20 \\ 9 \\ 20 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

Put your answers together: what is the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8)?

$$\vec{x}^T = [4 \quad 4 \quad 0 \quad 1 \quad 0]$$

3 Generalizing the example

Now let's generalize what happened in the example above. Consider the generic canonical form LP (CF) on page 1. Suppose it has m equality constraints and n decision variables (e.g. A has m rows and n columns). Assume that $m \leq n$ and $\text{rank}(A) = m$.

Suppose $\mathbf{x}$ is a basic solution. How many linearly independent constraints must be active at $\mathbf{x}$?

At least n

Since $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, how many nonnegativity bounds must be active?

At least $n-m$

Generalizing our observations from the example, we have the following theorem.

Theorem. Suppose $\mathbf{x}$ is a basic solution of a canonical form LP. Then there exists a set of m variables of $\mathbf{x}$ such that

(a) the columns of A corresponding to these m variables are linearly independent;
(b) the other $n - m$ variables are equal to 0.

Check your understanding of this theorem: back in the example, what is n and what is m? Which variables correspond to m linearly independent columns of A? Which $n - m$ variables are equal to 0?

$n = 5$, $m = 3$, $n - m = 2$

x, y, s_2 correspond to the chosen m LI columns of A

s_1, s_3 are the $n-m$ variables equal to 0

Let B be the submatrix of A corresponding to these m variables, and let $\mathbf{x}_B$ be the vector of these m variables. Since the columns of B are linearly independent, these variables are the unique solution to the system $B\mathbf{x}_B = \mathbf{b}$, like $(**)$ in the example. These m variables are potentially nonzero, while the other $n - m$ variables are forced to be zero.

Given a basic solution $\mathbf{x}$ of a canonical form LP:

- a variable is **basic** if it corresponds to one of the m linearly independent columns of A defining $\mathbf{x}$,
- a variable is **nonbasic** if it is not basic,
- the set of basic variables is referred to as the **basis** of $\mathbf{x}$.

In the basic solution $\mathbf{x}$ you computed on page 2, which variables are basic?

x, y, s_2

Which variables are nonbasic?

s_1, s_3

What is the basis?

$\{x, y, s_2\}$