Lesson 23. Basic Solutions in Canonical Form LPs

1 Overview

Recall that an LP in **canonical form** looks like this:

$$\begin{aligned} \text{minimize / maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{CF}$$

Note that

(a) all the general constraints are equalities, and
(b) all the decision variables are constrained to be nonnegative.

Also, recall that a solution $\mathbf{x}$ of an LP with n decision variables is a **basic solution** if

(a) it satisfies all equality constraints, and
(b) at least n constraints are active at $\mathbf{x}$ and are linearly independent.

The solution $\mathbf{x}$ is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of the LP.

Today, we will investigate what basic solutions look like for canonical form LPs.

2 Example

Consider the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 3x + 8y \\ \text{subject to} \quad & x + 4y + s_1 = 20 & (1) \\ & x + y + s_2 = 9 & (2) \\ & 2x + 3y + s_3 = 20 & (3) \\ & x \geq 0 & (4) \\ & y \geq 0 & (5) \\ & s_1 \geq 0 & (6) \\ & s_2 \geq 0 & (7) \\ & s_3 \geq 0 & (8) \end{aligned}$$

Identify the matrix A and the vectors $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{b}$ in the above canonical form LP. *Hint.* Let $\mathbf{x} = (x, y, s_1, s_2, s_3)$.

Suppose **x** is a basic solution. How many linearly independent constraints must be active at **x**? How many of these must be equality constraints, and how many of these must be nonnegativity bounds?

Verify that the constraints (1), (2), (3), (6), and (8) are linearly independent. (Use a calculator or MATLAB.)

Let's compute the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8). Since the basic solution is active at the nonnegativity bounds (6) and (8), s_1 and s_3 are "forced" to be zero. The other variables, x, y, and s_2 are potentially nonzero.

Substituting $s_1 = 0$ and $s_3 = 0$ into the other constraints (1), (2), and (3), we get:

$$\begin{aligned} x + 4y + (0) \qquad\qquad &= 20 \\ x + \ \ y \qquad + s_2 \qquad &= 9 \\ 2x + 3y \qquad\qquad + (0) &= 20 \end{aligned} \tag{$*$}$$

Let B be the submatrix of A consisting of columns corresponding to variables x, y, and s_2:

$$B = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

Verify that the columns of B linearly independent. (Use a calculator or MATLAB.)

Note that $(*)$ can be written as

$$B \begin{pmatrix} x \\ y \\ s_2 \end{pmatrix} = \mathbf{b} \qquad (**)$$

Use B^{-1} (it exists, right?) to find the values of x, y, and s_2. (Use a calculator or MATLAB.)

Put your answers together: what is the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8)?

3 Generalizing the example

Now let's generalize what happened in the example above. Consider the generic canonical form LP (CF) on page 1. Suppose it has m equality constraints and n decision variables (e.g. A has m rows and n columns). Assume that $m \leq n$ and $\text{rank}(A) = m$.

Suppose $\mathbf{x}$ is a basic solution. How many linearly independent constraints must be active at $\mathbf{x}$?

Since $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, how many nonnegativity bounds must be active?

Generalizing our observations from the example, we have the following theorem.

Theorem. Suppose $\mathbf{x}$ is a basic solution of a canonical form LP. Then there exists a set of m variables of $\mathbf{x}$ such that

(a) the columns of A corresponding to these m variables are linearly independent;
(b) the other $n - m$ variables are equal to 0.

Check your understanding of this theorem: back in the example, what is n and what is m? Which variables correspond to m linearly independent columns of A? Which $n - m$ variables are equal to 0?

Let B be the submatrix of A corresponding to these m variables, and let $\mathbf{x}_B$ be the vector of these m variables. Since the columns of B are linearly independent, these variables are the unique solution to the system $B\mathbf{x}_B = \mathbf{b}$, like $(**)$ in the example. These m variables are potentially nonzero, while the other $n - m$ variables are forced to be zero.

Given a basic solution $\mathbf{x}$ of a canonical form LP:

- a variable is **basic** if it corresponds to one of the m linearly independent columns of A defining $\mathbf{x}$,
- a variable is **nonbasic** if it is not basic,
- the set of basic variables is referred to as the **basis** of $\mathbf{x}$.

In the basic solution $\mathbf{x}$ you computed on page 2, which variables are basic?

Which variables are nonbasic?

What is the basis?