SA305 – Linear Programming Spring 2014
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Lessons 24 + 25. The Simplex Method

1 Review

- Given an LP with n decision variables, a solution $\mathbf{x}$ is **basic** if:
  (a) it satisfies all equality constraints
  (b) at least n linearly independent constraints are active at $\mathbf{x}$
- A **basic feasible solution (BFS)** is a basic solution that satisfies all constraints of the LP
- Canonical form LP:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

  - m equality constraints and n decision variables (e.g. A has m rows and n columns).
  - Standard assumptions: $m \leq n$, $\operatorname{rank}(A) = m$
- If $\mathbf{x}$ is a basic solution of a canonical form LP, there exist m **basic variables** of $\mathbf{x}$ such that
  (a) the columns of A corresponding to these m variables are linearly independent
  (b) the other $n - m$ **nonbasic variables** are equal to 0
- The set of basic variables is the **basis** of $\mathbf{x}$

2 Overview

- General improving search algorithm

```
1 Find an initial feasible solution x^0
2 Set t = 0
3 while x^t is not locally optimal do
4    Determine a simultaneously improving and feasible direction d at x^t
5    Determine step size λ
6    Compute new feasible solution x^{t+1} = x^t + λd
7    Set t = t + 1
8 end while
```

- The **simplex method** is a specialized version of improving search
  - For canonical form LPs
  - Starts at a BFS in Step 1
  - Considers directions that point towards other BFSes in Step 4
  - Takes the maximum possible step size in Step 5

Example 1. Throughout this lesson, we will use the canonical form LP below:

$$
\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 13x & + & 5y & & & & & & & & \\
\text{subject to} & 4x & + & y & + & s_1 & & & & & = & 24 \\
& x & + & 3y & & & + & s_2 & & & = & 24 \\
& 3x & + & 2y & & & & & + & s_3 & = & 23 \\
& x, & & y, & & s_1, & & s_2, & & s_3 & \geq & 0
\end{array}
$$

3 Initial solutions

- For now, we will start by guessing an initial BFS

Example 2. Verify that $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ is a BFS with basis $\mathcal{B}^0 = \{s_1, s_2, s_3\}$.

4 Finding feasible directions

- Two BFSes are **adjacent** if their bases differ by exactly 1 variable
- Suppose $\mathbf{x}^t$ is the current BFS with basis $\mathcal{B}^t$
- Approach: consider directions that point towards BFSes adjacent to $\mathbf{x}^t$
- To get a BFS adjacent to $\mathbf{x}^t$:
    - Put one nonbasic variable into $\mathcal{B}^t$
    - Take one basic variable out of $\mathcal{B}^t$
- Suppose we want to put nonbasic variable y into $\mathcal{B}^t$
- This corresponds to the **simplex direction $\mathbf{d}^y$** corresponding to nonbasic variable y
- $\mathbf{d}^y$ has a component for every decision variable
    - e.g. $\mathbf{d}^y = (d_x^y, d_y^y, d_{s_1}^y, d_{s_2}^y, d_{s_3}^y)$ for the LP in Example 1

- The components of the simplex direction $\mathbf{d}^y$ corresponding to nonbasic variable y are:
  - $d_y^y = 1$
  - $d_z^y = 0$ for all other nonbasic variables z
  - d_w^y (uniquely) determined by $A\mathbf{d} = \mathbf{0}$ for all basic variables w
- Why does this work? Remember for LPs, $\mathbf{d}$ is a feasible direction at $\mathbf{x}$ if

$$\mathbf{a}^\top \mathbf{d} \left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} 0 \quad \text{for each active constraint of the form} \quad \mathbf{a}^\top \mathbf{x} \left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} b$$

- Each nonbasic variable has a corresponding simplex direction

Example 3. The basis of the BFS $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ is $\mathcal{B}^0 = \{s_1, s_2, s_3\}$. For each nonbasic variable, x and y, we have a corresponding simplex direction. Compute the simplex directions $\mathbf{d}^x$ and $\mathbf{d}^y$.

5 Finding improving directions

- Once we've computed the simplex direction for each nonbasic variable, which one do we choose?
- We choose a simplex direction $\mathbf{d}$ that is improving
- Recall that if $f(\mathbf{x})$ is the objective function, $\mathbf{d}$ is an improving direction at $\mathbf{x}$ if

$$\nabla f(\mathbf{x})^\top \mathbf{d} \quad \begin{cases} > 0 & \text{when maximizing } f \\ < 0 & \text{when minimizing } f \end{cases}$$

- For LPs, $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$, and so $\nabla f(\mathbf{x}) = \mathbf{c}$ for any $\mathbf{x}$
- The **reduced cost** associated with nonbasic variable y is

$$\bar{c}_y = \mathbf{c}^\top \mathbf{d}^y$$

  where $\mathbf{d}^y$ is the simplex direction associated with y
- The simplex direction $\mathbf{d}^y$ associated with nonbasic variable y is improving if

$$\bar{c}_y \quad \begin{cases} > 0 & \text{for a maximization LP} \\ < 0 & \text{for a minimization LP} \end{cases}$$

Example 4. Consider the BFS $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ with basis $\mathcal{B}^0 = \{s_1, s_2, s_3\}$. Compute the reduced costs $\bar{c}_x$ and $\bar{c}_y$ for nonbasic variables x and y, respectively. Are $\mathbf{d}^x$ and $\mathbf{d}^y$ improving?

- If there is an improving simplex direction, we choose it
- If there is more than 1 improving simplex direction, we can choose any one of them
  - One option – **Dantzig's rule**: choose the improving simplex direction with the most improving reduced cost (maximization LP – most positive, minimization LP – most negative)
- **If there are no improving simplex directions, then the current BFS is a global optimal solution**

6 Determining the maximum step size

- We've picked an improving simplex direction – how far can we go in that direction?
- Suppose $\mathbf{x}^t$ is our current BFS, $\mathbf{d}$ is the improving simplex direction we chose
- Our next solution is $\mathbf{x}^t + \lambda\mathbf{d}$ for some value of $\lambda \geq 0$
- How big can we make λ while still remaining feasible?
- Recall that we computed $\mathbf{d}$ so that $A\mathbf{d} = \mathbf{0}$
- $\mathbf{x}^t + \lambda\mathbf{d}$ satisfies the equality constraints $A\mathbf{x} = \mathbf{b}$ no matter how large λ gets, since

$$A(\mathbf{x}^t + \lambda\mathbf{d}) = A\mathbf{x}^t + \lambda A\mathbf{d} = A\mathbf{x}^t = \mathbf{b}$$

- So, the only thing that can go wrong are the nonnegativity constraints

  ⇒ What is the largest λ such that $\mathbf{x}^t + \lambda\mathbf{d} \geq \mathbf{0}$?

Example 5. Suppose we choose the improving simplex direction $\mathbf{d}^x = (1, 0, -4, -1, 3)$. Compute the maximum step size λ for which $\mathbf{x}^0 + \lambda\mathbf{d}^x$ remains feasible.

- Note that only negative components of $\mathbf{d}$ determine maximum step size:

$$x_j + \lambda d_j \overset{?}{\geq} 0$$

- The **minimum ratio test**: starting at the BFS $\mathbf{x}$, if any component of the improving simplex direction $\mathbf{d}$ is negative, then the maximum step size is

$$\lambda_{\max} = \min \left\{ \frac{x_j}{-d_j} : d_j < 0 \right\}$$

Example 6. Verify that the minimum ratio test yields the same maximum step size you found in Example 5.

- What if $\mathbf{d}$ has no negative components?
- For example:
  - Suppose $\mathbf{x}^0 = (0, 0, 1, 2, 3)$ is a BFS
  - $\mathbf{d} = (1, 0, 2, 4, 3)$ is an improving simplex direction at $\mathbf{x}$
  - Then the next solution is

$$\mathbf{x}^0 + \lambda \mathbf{d} = (\lambda, 0, 1 + 2\lambda, 2 + 4\lambda, 3 + 3\lambda) \quad \text{for some value of } \lambda \geq 0$$

  - $\mathbf{x}^0 + \lambda \mathbf{d} \geq 0$ for all $\lambda \geq 0$!
  - We can improve our objective function and remain feasible forever!

  $\Rightarrow$ The LP is unbounded

- **Test for unbounded LPs**: if all components of an improving simplex direction are nonnegative, then the LP is unbounded

7 Updating the basis

- We have our improving simplex direction $\mathbf{d}$ and step size $\lambda_{\max}$
- We can compute our new solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max}\mathbf{d}$
- We also update the basis: update the set of basic variables
- **Entering and leaving variables**
  - The nonbasic variable corresponding to the chosen simplex direction enters the basis and becomes basic: this is the **entering variable**
  - Any one of the basic variables that define the maximum step size leaves the basis and becomes nonbasic: this is the **leaving variable**

Example 7. Compute $\mathbf{x}^1$. What is the basis $\mathcal{B}^1$ of $\mathbf{x}^1$?

8 Putting it all together: the simplex method

Step 0: Initialization. Identify a BFS $\mathbf{x}^0$. Set solution index $t = 0$.

Step 1: Simplex directions. For each nonbasic variable y, compute the corresponding simplex direction $\mathbf{d}^y$ and its reduced cost $\bar{c}_y$.

Step 2: Check for optimality. If no simplex direction is improving, stop. The current solution $\mathbf{x}^t$ is optimal. Otherwise, choose any improving simplex direction $\mathbf{d}$. Let x_e denote the entering variable.

Step 3: Step size. If $\mathbf{d} \geq \mathbf{0}$, stop. The LP is unbounded. Otherwise, choose the leaving variable x_ℓ by computing the maximum step size $\lambda_{\max}$ according to the minimum ratio test.

Step 4: Update solution and basis. Compute the new solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max}\mathbf{d}$. Replace x_ℓ by x_e in the basis. Set $t = t + 1$. Go to Step 1.