 

# Lesson 29. Bounds and the Dual LP

## 1 Overview

- It is often useful to quickly generate lower and upper bounds on the optimal value of an LP
- Many algorithms for optimization problems that consider LP "subproblems" rely on this
- How can we do this?

## 2 Finding lower bounds

**Example 1.** Consider the following LP:

$$\begin{aligned} z^* = \text{maximize} \quad & 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 5x_3 \leq 18 && (1) \\ & 5x_1 + 4x_2 + 3x_3 \leq 16 && (2) \\ & x_1, x_2, x_3 \geq 0 && (3) \end{aligned}$$

Denote the optimal value of this LP by $z^*$. Give a feasible solution to this LP and its value. How does this value compare to $z^*$?

- **For a maximization LP, any feasible solution gives a lower bound on the optimal value**
- We want the highest lower bound possible (i.e. the lower bound closest to the optimal value)

## 3 Finding upper bounds

- We want the lowest upper bound possible (i.e. the upper bound closest to the optimal value)
- For the LP in Example 1, we can show that the optimal value $z^*$ is at most 27
  - Any feasible solution $(x_1, x_2, x_3)$ must satisfy constraint (1)
  - ⇒ Any feasible solution $(x_1, x_2, x_3)$ must also satisfy constraint (1) multiplied by 3/2 on both sides:
  - The nonnegativity bounds (3) imply that any feasible solution $(x_1, x_2, x_3)$ must satisfy
  - Therefore, any feasible solution, including the optimal solution, must have value at most 27
- We can do better: we can show $z^* \leq 25$:
  - Any feasible solution $(x_1, x_2, x_3)$ must satisfy constraints (1) and (2)
  - ⇒ Any feasible solution $(x_1, x_2, x_3)$ must also satisfy $\left(\frac{1}{2} \times \text{constraint (1)}\right) + \text{constraint (2)}$:
  - The nonnegativity bounds (3) then imply that any feasible solution $(x_1, x_2, x_3)$ must satisfy

**Example 2.** Combine the constraints (1) and (2) of the LP in Example 1 to find a better upper bound on $z^*$ than 25.

- Let's generalize this process of combining constraints
- Let $y_1$ be the "multiplier" for constraint (1), and let $y_2$ be the "multiplier" for constraint (2)
- We require $y_1 \geq 0$ and $y_2 \geq 0$ so that multiplying constraints (1) and (2) by these values keeps the inequalities as "≤"
- We also want:

- Since we want the lowest upper bound, we want:

- Putting this all together, we can find the multipliers that find the best lower upper bound with the following LP!

$$\begin{aligned} \text{minimize} \quad & 18y_1 + 16y_2 \\ \text{subject to} \quad & 3y_1 + 5y_2 \geq 2 \\ & 2y_1 + 4y_2 \geq 3 \\ & 5y_1 + 3y_2 \geq 4 \\ & y_1 \geq 0,\ y_2 \geq 0 \end{aligned}$$

  - This is the **dual LP**, or simply the **dual** of the LP in Example 1
  - The LP in example is referred to as the **primal LP** or the **primal** – the original LP

## 4 In general...

- Every LP has a dual
- For minimization LPs
  - Any feasible solution gives an <u>upper bound</u> on the optimal value
  - One can construct a dual LP to give the greatest <u>lower bound</u> possible
- We can generalize the process we just went through to develop some mechanical rules to construct duals

## 5 Constructing the dual LP

0. Rewrite the primal so all variables are on the LHS and all constants are on the RHS
1. Assign each primal constraint a corresponding **dual variable** (multiplier)
2. Write the dual objective function
   - The objective function coefficient of a dual variable is the RHS coefficient of its corresponding primal constraint
   - The dual objective sense is the opposite of the primal objective sense
3. Write the dual constraint corresponding to each primal variable
   - The dual constraint LHS is found by looking at the coefficients of the corresponding primal variable ("go down the column")
   - The dual constraint RHS is the objective function coefficient of the corresponding primal variable
4. Use the **SOB rule** to determine dual variable bounds ($\geq 0$, $\leq 0$, free) and dual constraint comparisons ($\leq$, $\geq$, $=$)

| | max LP | $\leftrightarrow$ | min LP | |
|---|---|---|---|---|
| sensible | $\leq$ constraint | $\leftrightarrow$ | $y_i \geq 0$ | sensible |
| odd | $=$ constraint | $\leftrightarrow$ | $y_i$ free | odd |
| bizarre | $\geq$ constraint | $\leftrightarrow$ | $y_i \leq 0$ | bizarre |
| sensible | $x_i \geq 0$ | $\leftrightarrow$ | $\geq$ constraint | sensible |
| odd | $x_i$ free | $\leftrightarrow$ | $=$ constraint | odd |
| bizarre | $x_i \leq 0$ | $\leftrightarrow$ | $\leq$ constraint | bizarre |

**Example 3.** Take the dual of the following LP:

$$
\begin{aligned}
\text{minimize} \quad & 10x_1 + 9x_2 - 6x_3 \\
\text{subject to} \quad & 2x_1 - x_2 \geq 3 \\
& 5x_1 + 3x_2 - x_3 \leq 14 \\
& x_2 + x_3 = 1 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \geq 0
\end{aligned}
$$

- **The dual of the dual is the primal**
  - Try it with the dual you just found