Lesson 30. Weak and Strong Duality

1 Practice taking duals!

Example 1. State the dual of the following linear programs.

a.
$$\begin{array}{lll}
\text{minimize} & 5x_1 + x_2 - 4x_3 & \\
\text{subject to} & x_1 + x_2 + x_3 + x_4 = 19 & y_1 \;\; O \\
 & 4x_2 + 8x_4 \le 55 & y_2 \;\; B \\
 & x_1 + 6x_2 - x_3 \ge 7 & y_3 \;\; S \\
\underset{O}{x_1 \text{ free},} & \underset{S}{x_2 \ge 0}, \underset{S}{x_3 \ge 0}, \underset{B}{x_4 \le 0} &
\end{array}$$

b.
$$\begin{array}{lll}
\text{maximize} & 19y_1 + 4y_2 - 8z_2 & \\
\text{subject to} & 11y_1 + y_2 + z_1 = 15 & x_1 \;\; O \\
 & z_1 + 5z_2 \le 0 & x_2 \;\; S \\
 & y_1 - y_2 + z_2 \ge 4 & x_3 \;\; B \\
 & \underset{S}{y_1 \ge 0}, \underset{S}{y_2 \ge 0}, \underset{O}{z_1 \text{ free}}, \underset{O}{z_2 \text{ free}} &
\end{array}$$

a.
$$\begin{array}{lll}
\max & 19y_1 + 55y_2 + 7y_3 & \\
\text{s.t.} & y_1 + y_3 = 5 & x_1 \;\; O \\
 & y_1 + 4y_2 + 6y_3 \le 1 & x_2 \;\; S \\
 & y_1 - y_3 \le -4 & x_3 \;\; S \\
 & y_1 + 8y_2 \ge 0 & x_4 \;\; B \\
 & \underset{O}{y_1 \text{ free}}, \underset{B}{y_2 \le 0}, \underset{S}{y_3 \ge 0} &
\end{array}$$

b.
$$\begin{array}{lll}
\min & 15x_1 + 4x_3 & \\
\text{s.t.} & 11x_1 + x_3 \le 19 & y_1 \;\; S \\
 & x_1 - x_3 \le 0 & y_2 \;\; S \\
 & x_1 + x_2 = 0 & z_1 \;\; O \\
 & 5x_2 + x_3 = -8 & z_2 \;\; O \\
 & \underset{O}{x_1 \text{ free}}, \underset{S}{x_2 \ge 0}, \underset{B}{x_3 \le 0} &
\end{array}$$

2 Weak duality

- Consider the following primal-dual pair of LPs

$$\begin{array}{llll} \text{[P]} & \text{maximize} & \mathbf{c}^\top \mathbf{x} \\ & \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ & & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{llll} \text{[D]} & \text{minimize} & \mathbf{b}^\top \mathbf{y} \\ & \text{subject to} & A^\top \mathbf{y} \geq \mathbf{c} \\ & & \mathbf{y} \geq \mathbf{0} \end{array}$$

- Remember we constructed the dual in such a way that the multipliers $\mathbf{y}$ give us an upper bound on the optimal value of [P]

Weak Duality Theorem. Let $\mathbf{x}^*$ be a feasible solution to [P], and let $\mathbf{y}^*$ be a feasible solution to [D]. Then

$$\mathbf{c}^\top \mathbf{x}^* \leq \mathbf{b}^\top \mathbf{y}^*$$

Proof. $\vec{c}^T \vec{x}^* \leq (A^T \vec{y}^*)^T \vec{x}^* = \vec{y}^{*T} A \vec{x}^* \leq \vec{y}^{*T} \vec{b}$.

$\vec{x}^* \geq 0$, $\vec{y}^*$ feasible in [D]: $A^T \vec{y}^* \leq \vec{c}$

$\vec{y}^* \geq 0$, $\vec{x}^*$ feasible in [P]: $A\vec{x}^* \leq \vec{b}$

Corollary 1. If $\mathbf{x}^*$ is a feasible solution to [P], $\mathbf{y}^*$ is a feasible solution to [D], and

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$$

then (i) $\mathbf{x}^*$ is an optimal solution to [P] and (ii) $\mathbf{y}^*$ is an optimal solution to [D].

Proof.

(i) $\vec{c}^T \vec{x}^* = \vec{b}^T \vec{y}^* \geq \vec{c}^T \vec{x}$ for all feasible solutions $\vec{x}$ to [P]

Weak duality

(ii) $\vec{b}^T \vec{y}^* = \vec{c}^T \vec{x}^* \leq \vec{b}^T \vec{y}$ for all feasible solutions $\vec{y}$ to [D]

Weak duality

Corollary 2. If [P] is unbounded, then [D] must be infeasible.

Proof. By contradiction. Suppose [D] is feasible.
Let $\vec{y}^*$ be a feasible solution to [D].
Weak duality $\Rightarrow$ $\vec{c}^T\vec{x} \leq \vec{b}^T\vec{y}^*$ for all feasible solutions $\vec{x}$ to [P]
$\Rightarrow$ [P] cannot be unbounded, which is a contradiction.

Corollary 3. If [D] is unbounded, then [P] must be infeasible.

Proof. Similar to the previous corollary.

- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
  - See Rader p. 328 for an example
- All these theorems and corollaries apply to arbitrary primal-dual LP pairs, not just [P] and [D] above

3 Strong duality

Strong Duality Theorem. Let [P] denote a primal LP and [D] its dual.

a. If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.

b. If [P] and [D] both have feasible solutions, then
   - [P] has a finite optimal solution $\mathbf{x}^*$;
   - [D] has a finite optimal solution $\mathbf{y}^*$;
   - the optimal values of [P] and [D] are equal.

- This is an AMAZING fact
- Useful from theoretical, algorithmic, and modeling perspectives
- Even the simplex method implicitly uses duality: the reduced costs are essentially dual solutions that are infeasible until the last step