Lesson 30. Weak and Strong Duality

1 Practice taking duals!

Example 1. State the dual of the following linear programs.

a.
$$\begin{array}{ll} \text{minimize} & 5x_1 + x_2 - 4x_3 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 = 19 \\ & 4x_2 + 8x_4 \leq 55 \\ & x_1 + 6x_2 - x_3 \geq 7 \\ & x_2 \geq 0, x_3 \geq 0, x_4 \leq 0 \end{array}$$

b.
$$\begin{array}{ll} \text{maximize} & 19y_1 + 4y_2 - 8z_2 \\ \text{subject to} & 11y_1 + y_2 + z_1 = 15 \\ & z_1 + 5z_2 \leq 0 \\ & y_1 - y_2 + z_2 \geq 4 \\ & y_1 \geq 0, y_2 \geq 0 \end{array}$$

2 Weak duality

- Consider the following primal-dual pair of LPs

$$
\begin{array}{lll}
[\text{P}] & \text{maximize} & \mathbf{c}^\top \mathbf{x} \\
 & \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\
 & & \mathbf{x} \geq \mathbf{0}
\end{array}
\qquad\qquad
\begin{array}{lll}
[\text{D}] & \text{minimize} & \mathbf{b}^\top \mathbf{y} \\
 & \text{subject to} & A^\top \mathbf{y} \geq \mathbf{c} \\
 & & \mathbf{y} \geq \mathbf{0}
\end{array}
$$

- Remember we constructed the dual in such a way that the multipliers $\mathbf{y}$ give us an upper bound on the optimal value of [P]

Weak Duality Theorem. Let $\mathbf{x}^*$ be a feasible solution to [P], and let $\mathbf{y}^*$ be a feasible solution to [D]. Then

$$\mathbf{c}^\top \mathbf{x}^* \leq \mathbf{b}^\top \mathbf{y}^*$$

Corollary 1. If $\mathbf{x}^*$ is a feasible solution to [P], $\mathbf{y}^*$ is a feasible solution to [D], and

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$$

then (i) $\mathbf{x}^*$ is an optimal solution to [P] and (ii) $\mathbf{y}^*$ is an optimal solution to [D].

Corollary 2. If [P] is unbounded, then [D] must be infeasible.

Corollary 3. If [D] is unbounded, then [P] must be infeasible.

Proof. Similar to the previous corollary.

- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
  - See Rader p. 328 for an example
- All these theorems and corollaries apply to arbitrary primal-dual LP pairs, not just [P] and [D] above

3 Strong duality

Strong Duality Theorem. Let [P] denote a primal LP and [D] its dual.

a. If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.

b. If [P] and [D] both have feasible solutions, then
   - [P] has a finite optimal solution $\mathbf{x}^*$;
   - [D] has a finite optimal solution $\mathbf{y}^*$;
   - the optimal values of [P] and [D] are equal.

- This is an AMAZING fact
- Useful from theoretical, algorithmic, and modeling perspectives
- Even the simplex method implicitly uses duality: the reduced costs are essentially dual solutions that are infeasible until the last step