

Lesson 30. Weak and Strong Duality

1 Practice taking duals!

Example 1. State the dual of the following linear programs.

a. minimize $5x_1 + x_2 - 4x_3$
subject to $x_1 + x_2 + x_3 + x_4 = 19$
 $4x_2 + 8x_4 \leq 55$
 $x_1 + 6x_2 - x_3 \geq 7$
 $x_2 \geq 0, x_3 \geq 0, x_4 \leq 0$

b. maximize $19y_1 + 4y_2 - 8z_2$
subject to $11y_1 + y_2 + z_1 = 15$
 $z_1 + 5z_2 \leq 0$
 $y_1 - y_2 + z_2 \geq 4$
 $y_1 \geq 0, y_2 \geq 0$

2 Weak duality

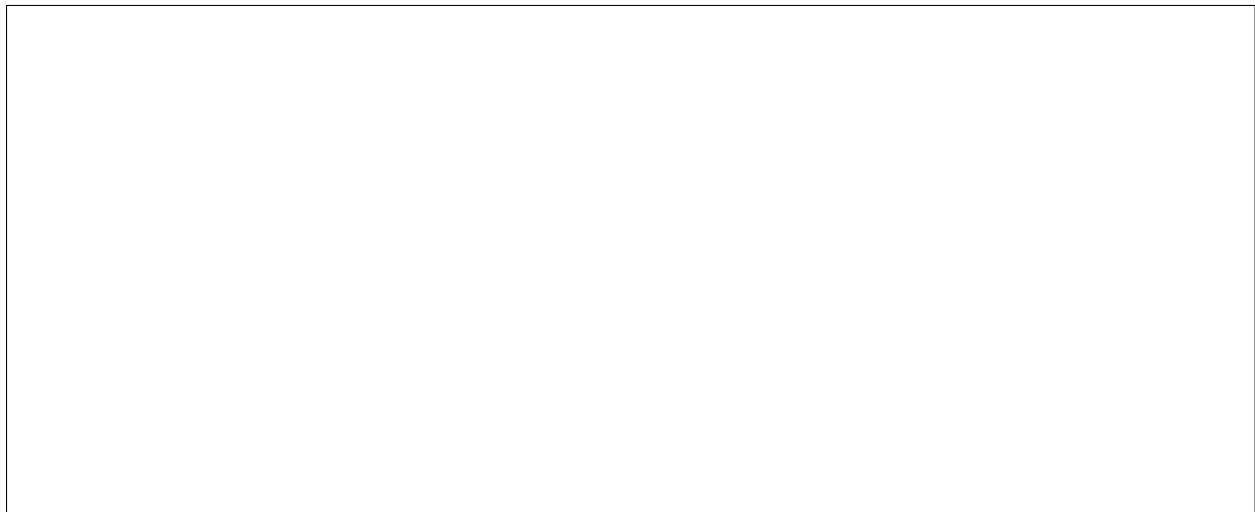
- Consider the following primal-dual pair of LPs

$$\begin{array}{ll} \text{[P]} & \text{maximize } \mathbf{c}^\top \mathbf{x} \\ & \text{subject to } A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{ll} \text{[D]} & \text{minimize } \mathbf{b}^\top \mathbf{y} \\ & \text{subject to } A^\top \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

- Remember we constructed the dual in such a way that the multipliers \mathbf{y} give us an upper bound on the optimal value of [P]

Weak Duality Theorem. Let \mathbf{x}^* be a feasible solution to [P], and let \mathbf{y}^* be a feasible solution to [D]. Then

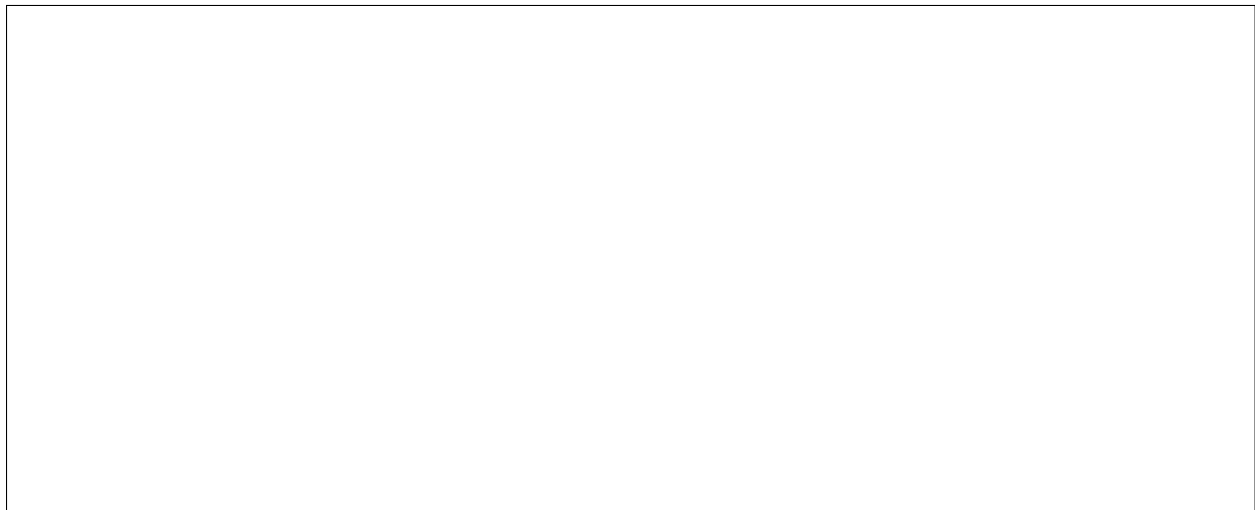
$$\mathbf{c}^\top \mathbf{x}^* \leq \mathbf{b}^\top \mathbf{y}^*$$



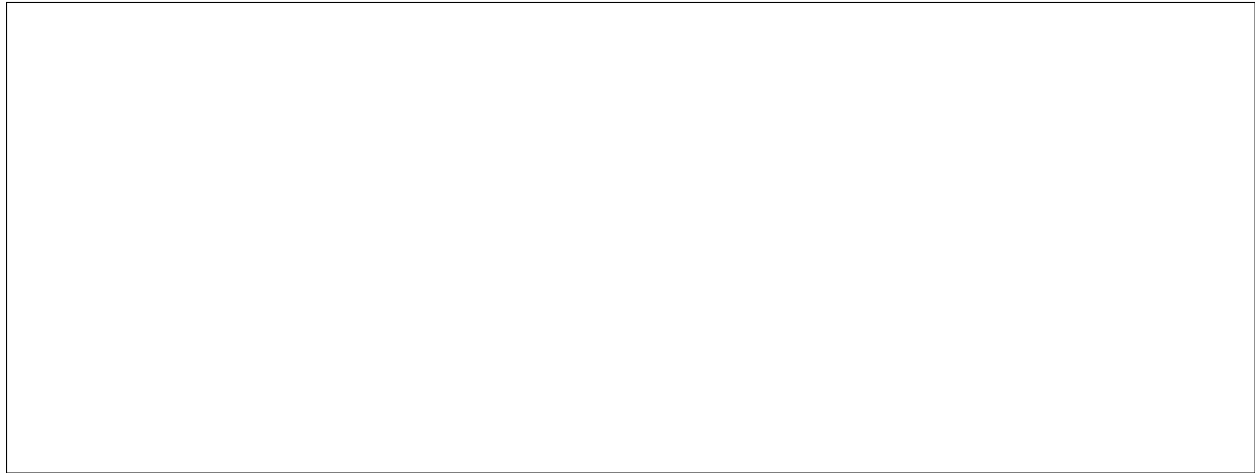
Corollary 1. If \mathbf{x}^* is a feasible solution to [P], \mathbf{y}^* is a feasible solution to [D], and

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$$

then (i) \mathbf{x}^* is an optimal solution to [P] and (ii) \mathbf{y}^* is an optimal solution to [D].



Corollary 2. If [P] is unbounded, then [D] must be infeasible.



Corollary 3. If [D] is unbounded, then [P] must be infeasible.

Proof. Similar to the previous corollary.

- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
 - See Rader p. 328 for an example
- All these theorems and corollaries apply to arbitrary primal-dual LP pairs, not just [P] and [D] above

3 Strong duality

Strong Duality Theorem. Let [P] denote a primal LP and [D] its dual.

- If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.
 - If [P] and [D] both have feasible solutions, then
 - [P] has a finite optimal solution \mathbf{x}^* ;
 - [D] has a finite optimal solution \mathbf{y}^* ;
 - the optimal values of [P] and [D] are equal.
- This is an AMAZING fact
 - Useful from theoretical, algorithmic, and modeling perspectives
 - Even the simplex method implicitly uses duality: the reduced costs are essentially dual solutions that are infeasible until the last step