Lesson 31. An Economic Interpretation of LP Duality

Today

- An economic interpretation of duality
- Complementary slackness

Review: weak and strong duality

- **Weak duality theorem:** For any primal-dual pair of LPs,

$$\begin{pmatrix} \text{objective function value} \\ \text{of any feasible solution} \\ \text{to the } \underline{\text{maximizing}} \text{ LP} \end{pmatrix} \leq \begin{pmatrix} \text{objective function value} \\ \text{of any feasible solution} \\ \text{to the } \underline{\text{minimizing}} \text{ LP} \end{pmatrix}$$

- **Corollary.** If the primal and dual have feasible solutions with the same objective function value, then these solutions must be optimal for the primal and dual, respectively

- **Corollary.** For any primal-dual pair of LPs, if one of the LPs is unbounded, then the other must be infeasible

  - Note that the reverse doesn't always hold: if one of the LPs is infeasible, the other is not necessarily unbounded

- **Strong duality theorem:**

  1. If the primal LP has finite optimal value, then
     - the dual has finite optimal value, and
     - the primal and dual have the same optimal value
  2. If the primal and dual have feasible solutions, then
     - both LPs have finite optimal values, and
     - the primal and dual have the same optimal value

Warm up

The Fulkerson Furniture Company produces desks, tables, and chairs. Each type of furniture requires a certain amount of lumber, finishing, and carpentry:

Resource	Desk	Table	Chair	Available
Lumber (sq ft)	8	6	2	48
Finishing (hrs)	3	2	1	20
Carpentry (hrs)	2	2	1	8
Profit ($)	60	30	20	

Assume that all furniture produced is sold, and that fractional solutions are acceptable. Write a linear program to determine how much furniture Fulkerson should produce in order to maximize its profits.

- Decision variables:

- Fulkerson's LP:

Economic interpretation of the dual LP

- Suppose an entrepreneur wants to purchase all of Fulkerson's resources (lumber, finishing, carpentry)
- What prices should she offer for the resources that will entice Fulkerson to sell?
- Define decision variables:

$$\begin{aligned} y_1 &= \text{price of 1 sq. ft. lumber} \\ y_2 &= \text{price of 1 hour of finishing} \\ y_3 &= \text{price of 1 hour of carpentry} \end{aligned}$$

- To buy all of Fulkerson's resources, entrepreneur pays:

- Entrepreneur wants to minimize costs
- Entrepreneur needs to offer resource prices that will <u>entice Fulkerson to sell</u>

- One desk uses
  - 8 sq. ft. of lumber
  - 3 hours of finishing
  - 2 hours of carpentry
  - One desk has profit of \$60

⇒ Entrepreneur should pay <u>at least</u> \$60 for this combination of resources:

- One table uses
  - 6 sq. ft. of lumber
  - 2 hours of finishing
  - 2 hours of carpentry
  - One table has profit of \$30

⇒ Entrepreneur should pay <u>at least</u> \$30 for this combination of resources:

- One chair uses
  - 2 sq. ft. of lumber
  - 1 hours of finishing
  - 1 hours of carpentry
  - One chair has profit of \$20

⇒ Entrepreneur should pay <u>at least</u> \$20 for this combination of resources:

- Increasing the availability of the resources potentially <u>increases</u> the maximum profits Fulkerson can achieve

⇒ Entrepreneur should pay nonnegative amounts for each resource:

- Putting this all together, we get:

$$\begin{array}{llll} \min & 48y_1 + 20y_2 + 8y_3 & & \\ \text{s.t.} & 8y_1 + 3y_2 + 2y_3 \geq 60 & & (x_1\text{: desks}) \\ & 6y_1 + 2y_2 + 2y_3 \geq 30 & & (x_2\text{: tables}) \\ & 2y_1 + y_2 + y_3 \geq 20 & & (x_3\text{: chairs}) \\ & y_1, \quad y_2, \quad y_3 \geq 0 & & \end{array}$$

- This is the dual of Fulkerson's LP!
- In summary:
  - Optimal dual solution ⇔ "fair" prices for associated resources
  - Known as **marginal prices** or **shadow prices**
- Strong duality
  - ⇒ Company's maximum revenue from selling furniture = Entrepreneur's minimum cost of purchasing resources
  - Equilibrium under perfect competition: company makes no excess profits
- This kind of economic interpretation is trickier for LPs with different types of constraints and variable bounds

Complementary slackness

- Optimal solution to Fulkerson's LP: $x_1 = 4, \; x_2 = 0, \; x_3 = 0$
- Resources used:

  lumber: $32 < 48$ finishing: $12 < 20$ carpentry: $8 = 8$

- How much would you pay for an extra sq. ft. of lumber?

- How much would you pay for an extra hour of finishing?

- Resource not fully utilized in optimal solution
  - ⇒ marginal price = 0
- **Primal complementary slackness:** either
  - a primal constraint is active at a primal optimal solution, or
  - the corresponding dual variable at optimality = 0
- Same logic applies to the dual

- Dual constraints ⇔ Primal decision variables
- **Dual complementary slackness:** either
  - a primal decision variable at optimality = 0, or
  - the corresponding dual constraint is active in a dual optimal solution

If we have time...

Consider the following LP:

$$\begin{aligned} \text{minimize} \quad & 3x_1 - x_2 + 8x_3 \\ \text{subject to} \quad & -x_1 + 8x_3 \le 6 \\ & 5x_1 - 3x_2 + 9x_3 \ge -2 \\ & x_1 \ge 0, x_2 \le 0, x_3 \ge 0 \end{aligned}$$

1. Write the dual.
2. Find a feasible solution to the primal and the dual.
3. Give a lower and an upper bound on the optimal value of the above LP.