

## Lesson 22. The Simplex Method

### 0 Review

- Given an LP with  $n$  decision variables, a solution  $\mathbf{x}$  is **basic** if:
  - (a) it satisfies all equality constraints
  - (b) at least  $n$  linearly independent constraints are active at  $\mathbf{x}$
- A **basic feasible solution (BFS)** is a basic solution that satisfies all constraints of the LP
- **Canonical form LP:**

$$\begin{aligned} & \text{maximize} && \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && A\mathbf{x} = \mathbf{b} \\ & && \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- $m$  equality constraints and  $n$  decision variables (e.g.  $A$  has  $m$  rows and  $n$  columns).
  - Standard assumptions:  $m \leq n$ ,  $\text{rank}(A) = m$
- If  $\mathbf{x}$  is a basic solution of a canonical form LP, there exist  $m$  **basic variables** of  $\mathbf{x}$  such that
  - (a) the columns of  $A$  corresponding to these  $m$  variables are linearly independent
  - (b) the other  $n - m$  **nonbasic variables** are equal to 0
- The set of basic variables is the **basis** of  $\mathbf{x}$

### 1 Overview

- General improving search algorithm
  - 1 Find an initial feasible solution  $\mathbf{x}^0$
  - 2 Set  $t = 0$
  - 3 **while**  $\mathbf{x}^t$  is not locally optimal **do**
  - 4   Determine a simultaneously improving and feasible direction  $\mathbf{d}$  at  $\mathbf{x}^t$
  - 5   Determine step size  $\lambda$
  - 6   Compute new feasible solution  $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda \mathbf{d}$
  - 7   Set  $t = t + 1$
  - 8 **end while**
- The **simplex method** is a specialized version of improving search
  - For canonical form LPs
  - Starts at a BFS in Step 1
  - Considers directions that point towards other BFSes in Step 4
  - Takes the maximum possible step size in Step 5

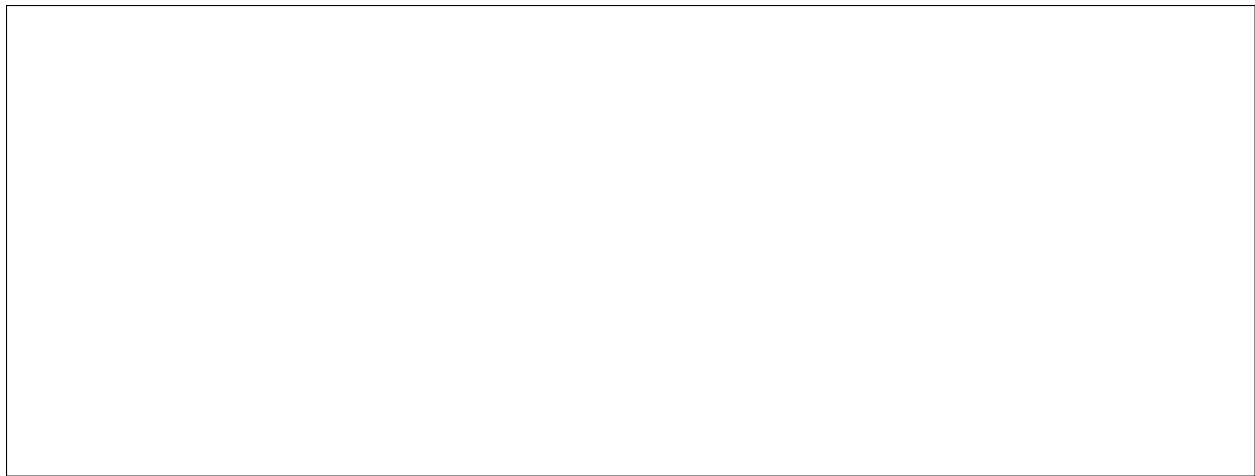
**Example 1.** Throughout this lesson, we will use the canonical form LP below:

$$\begin{aligned} & \text{maximize} && 13x + 5y \\ & \text{subject to} && 4x + y + s_1 = 24 \\ & && x + 3y + s_2 = 24 \\ & && 3x + 2y + s_3 = 23 \\ & && x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

## 2 Initial solutions

- For now, we will start by guessing an initial BFS

**Example 2.** Verify that  $\mathbf{x}^0 = (0, 0, 24, 24, 23)$  is a BFS with basis  $\mathcal{B}^0 = \{s_1, s_2, s_3\}$ .



## 3 Finding feasible directions

- Two BFSes are **adjacent** if their bases differ by exactly 1 variable
- Suppose  $\mathbf{x}^t$  is the current BFS with basis  $\mathcal{B}^t$
- Approach: consider directions that point towards BFSes adjacent to  $\mathbf{x}^t$
- To get a BFS adjacent to  $\mathbf{x}^t$ :
  - Put one nonbasic variable into  $\mathcal{B}^t$
  - Take one basic variable out of  $\mathcal{B}^t$
- Suppose we want to put nonbasic variable  $y$  into  $\mathcal{B}^t$
- This corresponds to the **simplex direction**  $\mathbf{d}^y$  corresponding to nonbasic variable  $y$
- $\mathbf{d}^y$  has a component for every decision variable
  - e.g.  $\mathbf{d}^y = (d_x^y, d_y^y, d_{s_1}^y, d_{s_2}^y, d_{s_3}^y)$  for the LP in Example 1

- The components of the simplex direction  $\mathbf{d}^y$  corresponding to nonbasic variable  $y$  are:
  - $d_y^y = 1$
  - $d_z^y = 0$  for all other nonbasic variables  $z$
  - $d_w^y$  (uniquely) determined by  $\mathbf{A}\mathbf{d} = \mathbf{0}$  for all basic variables  $w$
- Why does this work? Remember for LPs,  $\mathbf{d}$  is a feasible direction at  $\mathbf{x}$  if
  - $\mathbf{a}^\top \mathbf{d} \leq 0$  for each active constraint of the form  $\mathbf{a}^\top \mathbf{x} \leq b$
  - $\mathbf{a}^\top \mathbf{d} \geq 0$  for each active constraint of the form  $\mathbf{a}^\top \mathbf{x} \geq b$
  - $\mathbf{a}^\top \mathbf{d} = 0$  for each active constraint of the form  $\mathbf{a}^\top \mathbf{x} = b$
- Each nonbasic variable has a corresponding simplex direction

**Example 3.** The basis of the BFS  $\mathbf{x}^0 = (0, 0, 24, 24, 23)$  is  $\mathcal{B}^0 = \{s_1, s_2, s_3\}$ . For each nonbasic variable,  $x$  and  $y$ , we have a corresponding simplex direction. Compute the simplex directions  $\mathbf{d}^x$  and  $\mathbf{d}^y$ .

#### 4 Finding improving directions

- Once we've computed the simplex direction for each nonbasic variable, which one do we choose?
- We choose a simplex direction  $\mathbf{d}$  that is improving
- Recall that if  $f(\mathbf{x})$  is the objective function,  $\mathbf{d}$  is an improving direction at  $\mathbf{x}$  if

$$\nabla f(\mathbf{x})^\top \mathbf{d} \begin{cases} > 0 & \text{when maximizing } f \\ < 0 & \text{when minimizing } f \end{cases}$$

- For LPs,  $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$ , and so  $\nabla f(\mathbf{x}) = \mathbf{c}$  for any  $\mathbf{x}$
- The **reduced cost** associated with nonbasic variable  $y$  is

$$\bar{c}_y = \mathbf{c}^\top \mathbf{d}^y$$

where  $\mathbf{d}^y$  is the simplex direction associated with  $y$

- The simplex direction  $\mathbf{d}^y$  associated with nonbasic variable  $y$  is improving if

$$\bar{c}_y \begin{cases} > 0 & \text{for a maximization LP} \\ < 0 & \text{for a minimization LP} \end{cases}$$

**Example 4.** Consider the BFS  $\mathbf{x}^0 = (0, 0, 24, 24, 23)$  with basis  $\mathcal{B}^0 = \{s_1, s_2, s_3\}$ . Compute the reduced costs  $\bar{c}_x$  and  $\bar{c}_y$  for nonbasic variables  $x$  and  $y$ , respectively. Are  $\mathbf{d}^x$  and  $\mathbf{d}^y$  improving?

- If there is an improving simplex direction, we choose it
- If there is more than 1 improving simplex direction, we can choose any one of them
  - One option – **Dantzig’s rule**: choose the improving simplex direction with the most improving reduced cost (maximization LP – most positive, minimization LP – most negative)
- **If there are no improving simplex directions, then the current BFS is a global optimal solution**

## 5 Determining the maximum step size

- We’ve picked an improving simplex direction – how far can we go in that direction?
- Suppose  $\mathbf{x}^t$  is our current BFS,  $\mathbf{d}$  is the improving simplex direction we chose
- Our next solution is  $\mathbf{x}^t + \lambda \mathbf{d}$  for some value of  $\lambda \geq 0$
- How big can we make  $\lambda$  while still remaining feasible?
- Recall that we computed  $\mathbf{d}$  so that  $A\mathbf{d} = \mathbf{0}$
- $\mathbf{x}^t + \lambda \mathbf{d}$  satisfies the equality constraints  $A\mathbf{x} = \mathbf{b}$  no matter how large  $\lambda$  gets, since

$$A(\mathbf{x}^t + \lambda \mathbf{d}) = A\mathbf{x}^t + \lambda A\mathbf{d} = A\mathbf{x}^t = \mathbf{b}$$

- So, the only thing that can go wrong are the nonnegativity constraints
  - ⇒ What is the largest  $\lambda$  such that  $\mathbf{x}^t + \lambda \mathbf{d} \geq \mathbf{0}$ ?

**Example 5.** Suppose we choose the improving simplex direction  $\mathbf{d}^x = (1, 0, -4, -1, 3)$ . Compute the maximum step size  $\lambda$  for which  $\mathbf{x}^0 + \lambda\mathbf{d}^x$  remains feasible.

- Note that only negative components of  $\mathbf{d}$  determine maximum step size:

$$x_j + \lambda d_j \geq 0$$

- The **minimum ratio test**: starting at the BFS  $\mathbf{x}$ , if any component of the improving simplex direction  $\mathbf{d}$  is negative, then the maximum step size is

$$\lambda_{\max} = \min \left\{ \frac{x_j}{-d_j} : d_j < 0 \right\}$$

**Example 6.** Verify that the minimum ratio test yields the same maximum step size you found in Example 5.

- What if  $\mathbf{d}$  has no negative components?
- For example:
  - Suppose  $\mathbf{x}^0 = (0, 0, 1, 2, 3)$  is a BFS
  - $\mathbf{d} = (1, 0, 2, 4, 3)$  is an improving simplex direction at  $\mathbf{x}$
  - Then the next solution is

$$\mathbf{x}^0 + \lambda\mathbf{d} = (\lambda, 0, 1 + 2\lambda, 2 + 4\lambda, 3 + 3\lambda) \quad \text{for some value of } \lambda \geq 0$$

- $\mathbf{x}^0 + \lambda \mathbf{d} \geq 0$  for all  $\lambda \geq 0$ !
  - We can improve our objective function and remain feasible forever!
- ⇒ The LP is unbounded

- **Test for unbounded LPs:** if all components of an improving simplex direction are nonnegative, then the LP is unbounded

## 6 Updating the basis

- We have our improving simplex direction  $\mathbf{d}$  and step size  $\lambda_{\max}$
- We can compute our new solution  $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max} \mathbf{d}$
- We also update the basis: update the set of basic variables
- **Entering and leaving variables**
  - The nonbasic variable corresponding to the chosen simplex direction enters the basis and becomes basic: this is the **entering variable**
  - Any one of the basic variables that define the maximum step size leaves the basis and becomes nonbasic: this is the **leaving variable**

**Example 7.** Compute  $\mathbf{x}^1$ . What is the basis  $\mathcal{B}^1$  of  $\mathbf{x}^1$ ?

## 7 Putting it all together: the simplex method

**Step 0: Initialization.** Identify a BFS  $\mathbf{x}^0$ . Set solution index  $t = 0$ .

**Step 1: Simplex directions.** For each nonbasic variable  $y$ , compute the corresponding simplex direction  $\mathbf{d}^y$  and its reduced cost  $\bar{c}_y$ .

**Step 2: Check for optimality.** If no simplex direction is improving, stop. The current solution  $\mathbf{x}^t$  is optimal. Otherwise, choose any improving simplex direction  $\mathbf{d}$ . Let  $x_e$  denote the entering variable.

**Step 3: Step size.** If  $\mathbf{d} \geq \mathbf{0}$ , stop. The LP is unbounded. Otherwise, choose the leaving variable  $x_\ell$  by computing the maximum step size  $\lambda_{\max}$  according to the minimum ratio test.

**Step 4: Update solution and basis.** Compute the new solution  $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max} \mathbf{d}$ . Replace  $x_\ell$  with  $x_e$  in the basis. Set  $t = t + 1$ . Go to Step 1.