# Lesson 23. The Simplex Method – Example

**Problem 1.** Consider the following LP:

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 \leq 18 \\ & 4x_1 + 2x_2 + 5x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

a. Construct the canonical form of this LP.

b. Use the simplex method to solve the canonical form LP you wrote in part a. In particular:

- Construct your initial BFS and basis by making the nonslack variables having value 0.
- Choose your entering variable using **Dantzig's rule** – that is, choose the improving simplex direction with the most positive reduced cost. (If this was a minimization LP, you would choose the improving simplex direction with the most negative reduced cost.)

c. What is the optimal value of the canonical form LP you wrote in part a? Give an optimal solution.

d. What is the optimal value of the original LP above? Give an optimal solution.

a. $$\begin{aligned} \max \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + 4x_3 + s_1 = 18 \\ & 4x_1 + 2x_2 + 5x_3 + s_2 = 10 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

b+c. $\vec{x}^0 = (\overbrace{0, 0, 0}^{\text{nonslack} = 0}, 18, 10) \qquad \mathcal{B}^0 = \{s_1, s_2\}$

$\vec{d}^{x_1}$: $\vec{d}^{x_1} = (1, 0, 0, d_{s_1}, d_{s_2})$

$A\vec{d}^{x_1} = 0$: $\left.\begin{array}{l} 2 + d_{s_1} = 0 \\ 4 + d_{s_2} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} d_{s_1} = -2 \\ d_{s_2} = -4 \end{array}$

$\Rightarrow \vec{d}^{x_1} = (1, 0, 0, -2, -4)$

$\bar{c}_{x_1} = 4$

$\vec{d}^{x_2}$: $\vec{d}^{x_2} = (0, 1, 0, d_{s_1}, d_{s_2})$

$A\vec{d}^{x_2} = 0$: $\left.\begin{array}{l} -1 + d_{s_1} = 0 \\ 2 + d_{s_2} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} d_{s_1} = 1 \\ d_{s_2} = -2 \end{array}$

$\Rightarrow \vec{d}^{x_2} = (0, 1, 0, 1, -2)$

$\bar{c}_{x_2} = 3$

$\vec{d}^{x_3}$: $\vec{d}^{x_3} = (0, 0, 1, d_{s_1}, d_{s_2})$

$A\vec{d}^{x_3} = 0$: $\left.\begin{array}{l} 4 + d_{s_1} = 0 \\ 5 + d_{s_2} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} d_{s_1} = -4 \\ d_{s_2} = -5 \end{array}$

$\Rightarrow \vec{d}^{x_3} = (0, 0, 1, -4, -5)$

$\boxed{\bar{c}_{x_3} = 5}$ choose $x_3$ as entering

MRT: $\lambda_{max} = \min\left\{\underset{s_1}{\frac{18}{4}}, \underset{s_2}{\frac{10}{5}}\right\} = 2$ $\quad s_2$ is leaving

$\Rightarrow \vec{x}^1 = \vec{x}^0 + \lambda_{max}\vec{d}^{x_3} = (0, 0, 2, 10, 0)$

$\mathcal{B}^1 = \{x_3, s_1\}$

$\vec{x}^1 = (0, 0, 2, 10, 0) \qquad \mathcal{B}^1 = \{x_3, s_1\}$

$\underline{\vec{d}^{x_1}}$: $\vec{d}^{x_1} = (1, 0, d_{x_3}, d_{s_1}, 0)$

$A\vec{d}^{x_1} = 0$: $2 + 4d_{x_3} + d_{s_1} = 0$
$\qquad 4 + 5d_{x_3} = 0$

$\Rightarrow \vec{d}^{x_1} = (1, 0, -\frac{4}{5}, \frac{6}{5}, 0)$

$\bar{c}_{x_1} = 0$

$\underline{\vec{d}^{x_2}}$: $\vec{d}^{x_2} = (0, 1, d_{x_3}, d_{s_1}, 0)$

$A\vec{d}^{x_2} = 0$: $-1 + 4d_{x_3} + d_{s_1} = 0$
$\qquad 2 + 5d_{x_3} = 0$

$\Rightarrow \vec{d}^{x_2} = (0, 1, -\frac{2}{5}, \frac{13}{5}, 0)$

$\boxed{\bar{c}_{x_2} = 1}$ choose $x_2$ as entering

$\underline{\vec{d}^{s_2}}$: $\vec{d}^{s_2} = (0, 0, d_{x_3}, d_{s_1}, 1)$

$A\vec{d}^{s_2} = 0$: $4d_{x_3} + d_{s_1} = 0$
$\qquad 1 + 5d_{x_3} = 0$

$\Rightarrow \vec{d}^{s_2} = (0, 0, -\frac{1}{5}, \frac{4}{5}, 1)$

$\bar{c}_{s_2} = -1$

$\underline{\text{MRT}}$: $\lambda_{max} = \min\left\{\overset{x_3}{\frac{2}{2/5}}\right\} = 5 \qquad$ $x_3$ is leaving

$\Rightarrow \vec{x}^2 = \vec{x}^1 + \lambda_{max}\vec{d}^{x_2} = (0, 5, 0, 23, 0)$
$\mathcal{B}^2 = \{x_2, s_1\}$

$\vec{x}^2 = (0, 5, 0, 23, 0) \qquad \mathcal{B}^2 = \{x_2, s_1\}$

$\underline{\vec{d}^{x_1}}$: $\vec{d}^{x_1} = (1, d_{x_2}, 0, d_{s_1}, 0)$

$A\vec{d}^{x_1} = 0$: $2 - d_{x_2} + d_{s_1} = 0$
$\qquad 4 + 2d_{x_2} = 0$

$\Rightarrow \vec{d}^{x_1} = (1, -2, 0, -4, 0)$

$\bar{c}_{x_1} = -2$

$\underline{\vec{d}^{x_3}}$: $\vec{d}^{x_3} = (0, d_{x_2}, 1, d_{s_1}, 0)$

$A\vec{d}^{x_2} = 0$: $4 - d_{x_2} + d_{s_1} = 0$
$\qquad 5 + 2d_{x_2} = 0$

$\Rightarrow \vec{d}^{x_2} = (0, -\frac{5}{2}, 1, -\frac{13}{2}, 0)$

$\bar{c}_{x_2} = 1$

$\underline{\vec{d}^{s_2}}$: $\vec{d}^{s_2} = (0, d_{x_2}, 0, d_{s_1}, 1)$

$A\vec{d}^{s_2} = 0$: $-d_{x_2} + d_{s_1} = 0$
$\qquad 1 + 2d_{x_2} = 0$

$\Rightarrow \vec{d}^{s_2} = (0, -\frac{1}{2}, 0, -\frac{1}{2}, 1)$

$\bar{c}_{s_2} = -\frac{3}{2}$

$\Rightarrow$ No simplex directions are improving $\Rightarrow$ $\vec{x}^2$ is optimal, w/ value 15

d. In the original LP, $x_1 = 0$, $x_2 = 5$, $x_3 = 0$ is an optimal solution, w/ value 15.