

Lesson 26. Bounds and the Dual LP

1 Overview

- It is often useful to quickly generate lower and upper bounds on the optimal value of an LP
- Many algorithms for optimization problems that consider LP “subproblems” rely on this
- How can we do this?

2 Finding lower bounds

Example 1. Consider the following LP:

$$\begin{aligned} z^* = \text{maximize} \quad & 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 5x_3 \leq 18 & (1) \\ & 5x_1 + 4x_2 + 3x_3 \leq 16 & (2) \\ & x_1, x_2, x_3 \geq 0 & (3) \end{aligned}$$

Denote the optimal value of this LP by z^* . Give a feasible solution to this LP and its value. How does this value compare to z^* ?

- **For a maximization LP, any feasible solution gives a lower bound on the optimal value**
- We want the highest lower bound possible (i.e. the lower bound closest to the optimal value)

3 Finding upper bounds

- We want the lowest upper bound possible (i.e. the upper bound closest to the optimal value)
 - For the LP in Example 1, we can show that the optimal value z^* is at most 27
 - Any feasible solution (x_1, x_2, x_3) must satisfy constraint (1)
- ⇒ Any feasible solution (x_1, x_2, x_3) must also satisfy constraint (1) multiplied by $3/2$ on both sides:

- The nonnegativity bounds (3) imply that any feasible solution (x_1, x_2, x_3) must satisfy

- Therefore, any feasible solution, including the optimal solution, must have value at most 27

- We can do better: we can show $z^* \leq 25$:

- Any feasible solution (x_1, x_2, x_3) must satisfy constraints (1) and (2)

⇒ Any feasible solution (x_1, x_2, x_3) must also satisfy $\left(\frac{1}{2} \times \text{constraint (1)}\right) + \text{constraint (2)}$:

- The nonnegativity bounds (3) then imply that any feasible solution (x_1, x_2, x_3) must satisfy

Example 2. Combine the constraints (1) and (2) of the LP in Example 1 to find a better upper bound on z^* than 25.

- Let's generalize this process of combining constraints
- Let y_1 be the “multiplier” for constraint (1), and let y_2 be the “multiplier” for constraint (2)
- We require $y_1 \geq 0$ and $y_2 \geq 0$ so that multiplying constraints (1) and (2) by these values keeps the inequalities as “ \leq ”
- We also want:

- Since we want the lowest upper bound, we want:

- Putting this all together, we can find the multipliers that find the best lower upper bound with the following LP!

$$\begin{aligned}
 &\text{minimize} && 18y_1 + 16y_2 \\
 &\text{subject to} && 3y_1 + 5y_2 \geq 2 \\
 &&& 2y_1 + 4y_2 \geq 3 \\
 &&& 5y_1 + 3y_2 \geq 4 \\
 &&& y_1 \geq 0, y_2 \geq 0
 \end{aligned}$$

- This is the **dual LP**, or simply the **dual** of the LP in Example 1
- The LP in example is referred to as the **primal LP** or the **primal** – the original LP

4 In general...

- Every LP has a dual
- For minimization LPs
 - Any feasible solution gives an upper bound on the optimal value
 - One can construct a dual LP to give the greatest lower bound possible
- We can generalize the process we just went through to develop some mechanical rules to construct duals

5 Constructing the dual LP

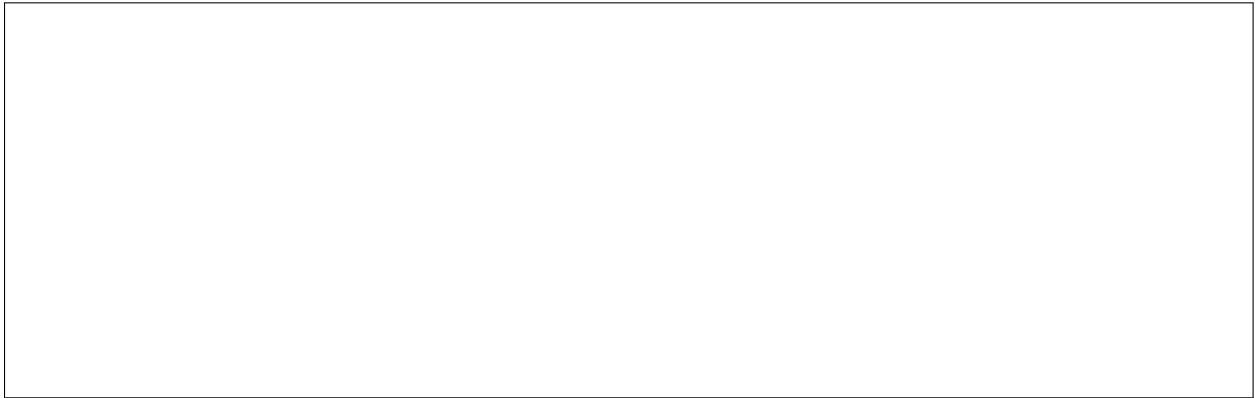
0. Rewrite the primal so all variables are on the LHS and all constants are on the RHS
1. Assign each primal constraint a corresponding **dual variable** (multiplier)
2. Write the dual objective function
 - The objective function coefficient of a dual variable is the RHS coefficient of its corresponding primal constraint
 - The dual objective sense is the opposite of the primal objective sense
3. Write the dual constraint corresponding to each primal variable
 - The dual constraint LHS is found by looking at the coefficients of the corresponding primal variable (“go down the column”)
 - The dual constraint RHS is the objective function coefficient of the corresponding primal variable
4. Use the **SOB rule** to determine dual variable bounds (≥ 0 , ≤ 0 , free) and dual constraint comparisons (\leq , \geq , $=$)

	max LP	\leftrightarrow	min LP	
sensible	\leq constraint	\leftrightarrow	$y_i \geq 0$	sensible
odd	$=$ constraint	\leftrightarrow	y_i free	odd
bizarre	\geq constraint	\leftrightarrow	$y_i \leq 0$	bizarre
sensible	$x_i \geq 0$	\leftrightarrow	\geq constraint	sensible
odd	x_i free	\leftrightarrow	$=$ constraint	odd
bizarre	$x_i \leq 0$	\leftrightarrow	\leq constraint	bizarre

Example 3. Take the dual of the following LP:

$$\begin{aligned}
 &\text{minimize} && 10x_1 + 9x_2 - 6x_3 \\
 &\text{subject to} && 2x_1 - x_2 \geq 3 \\
 &&& 5x_1 + 3x_2 - x_3 \leq 14 \\
 &&& x_2 + x_3 = 1 \\
 &&& x_1 \geq 0, x_2 \leq 0, x_3 \geq 0
 \end{aligned}$$

Example 4. Take the dual of the dual LP you found in Example 3.



- In general, **the dual of the dual is the primal**

6 Up next...

- Duality theorems: relationships between the primal and dual LPs