SA305 – Linear Programming | Spring 2015
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Lesson 29. Maximin and Minimax Objectives

1 The minimum of a collection of functions

Example 1. Santa Claus is trying to decide how to give candy canes to three children: Ann, Bob, and Carol. Because Santa is a very busy person, he has decided to give the same number of candy canes to each child. Let x be the number of candy canes each child receives. Also, because Santa knows everything, he knows the happiness level of each child as a function of the number of candy canes he or she receives:

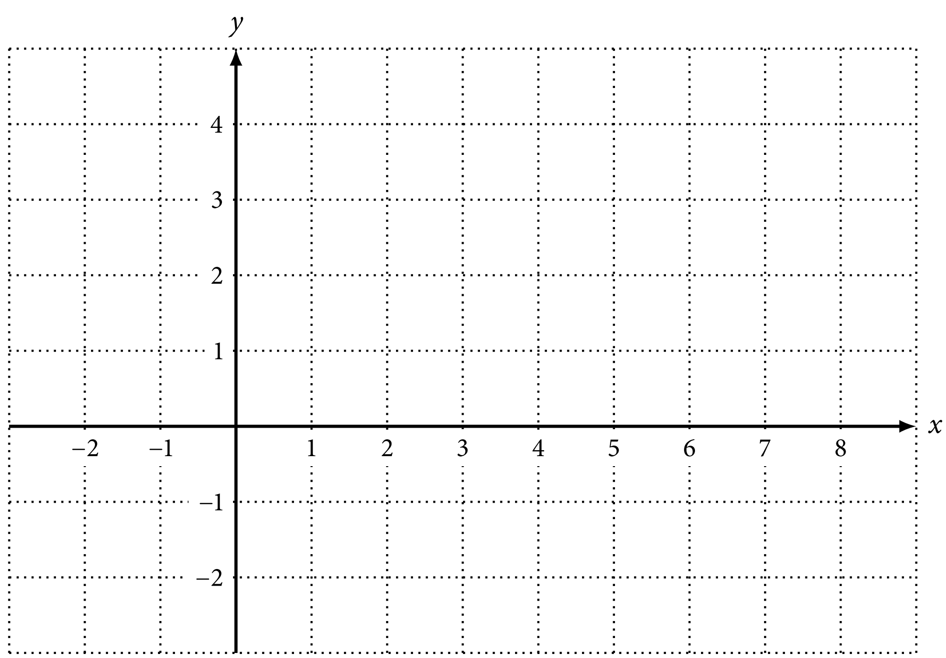
$$\text{Ann:} \quad 1 + 2x \qquad \text{Bob:} \quad 2 + x \qquad \text{Carol:} \quad 5 - \frac{1}{2}x$$

Due to the struggling economy, Santa's budget limits him to give each child at most 6 candy canes. To be fair to all 3 children, he has decided that he wants to **maximize the minimum happiness level of all 3 children**. In other words, he is trying to maximize the worst-case happiness level.

Let $f(x)$ be the minimum happiness level of all 3 children when each child receives x candy canes:

What is $f(0)$? $f(1)$? $f(2)$?

Graph $f(x)$:



Santa's optimization problem is:

By looking at the graph of $f(x)$, give an optimal solution to Santa's optimization problem. What are Ann's, Bob's, and Carol's happiness levels at this solution?

Observation. The minimum of a collection of numbers is the largest value that is less than or equal to each number in the collection.

For example, consider $\min\{3, 8, -2, 6, 9\}$.

Using this observation, we can rewrite Santa's optimization problem as:

This looks familiar...

What if we maximized the sum of the happiness factors of all 3 children? What is the optimal solution? What is Ann's, Bob's, and Carol's happiness levels at this solution?

$\Longrightarrow$ **Maximizing the minimum results in more uniform performance than maximizing the sum**

2 Maximin objective functions

- Define:
  - decision variables x_j for $j \in \{1, \ldots, n\}$
  - constants a_{ij} for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$
  - constants b_i for $i \in \{1, \ldots, m\}$
- Consider a **maximin objective function**:

$$\text{maximize} \quad \min \left\{ \sum_{j=1}^{n} a_{1j}x_j + b_1, \;\; \sum_{j=1}^{n} a_{2j}x_j + b_2, \;\; \ldots, \;\; \sum_{j=1}^{n} a_{mj}x_j + b_m \right\}$$

- We can convert a maximin objective function into a linear objective function and linear constraints:
  - Add auxiliary decision variable z
  - Change objective:

$$\text{maximize} \quad z$$

  - Add constraints:

$$z \leq \sum_{j=1}^{n} a_{ij}x_j + b_i \qquad \text{for } i \in \{1, \ldots, m\}$$

- Main idea: the minimum of a collection of numbers is <u>the largest value that is less than or equal to each number in the collection</u>.

3 Minimax objective functions

- We can similarly convert a **minimax objective function**

$$\text{minimize} \quad \max \left\{ \sum_{j=1}^{n} a_{1j}x_j + b_1, \;\; \sum_{j=1}^{n} a_{2j}x_j + b_2, \;\; \ldots, \;\; \sum_{j=1}^{n} a_{mj}x_j + b_m \right\}$$

  into a linear objective function and constraints:
  - Add auxiliary decision variable z
  - Change objective:

$$\text{minimize} \quad z$$

  - Add constraints:

$$z \geq \sum_{j=1}^{n} a_{ij}x_j + b_i \qquad \text{for } i \in \{1, \ldots, m\}$$

- Similar idea: the maximum of a collection of numbers is <u>the smallest value that is greater than or equal to each number in the collection</u>.

Example 2. The State of Simplex wants to divide the effort of its on-duty officers among 8 highway segments to reduce speeding incidents. You, the analyst, were able to estimate that for each highway segment $j \in \{1, \dots, 8\}$, the weekly reduction in speeding incidents is $r_j + s_j x_j$, where x_j is the number of officers assigned to segment j. Due to local ordinances, there is an upper bound u_j on the number of officers assigned to highway segment j per week, for $j \in \{1, \dots, 8\}$. There are 25 officers per week to allocate.

The State of Simplex has decided that it wants to maximize the worst-case reduction in speeding incidents among all highway segments. Write a linear program that allocates officers to highway segments according to this objective.

Input parameters.

$$
\begin{aligned}
H &= \text{set of highway segments} = \{1, \dots, 8\} \\
r_j, s_j &= \text{coefficients on weekly incident reduction function for highway segment } j && \text{for } j \in H \\
u_j &= \text{upper bound on number of officers assigned to highway segment } j \text{ per week} && \text{for } j \in H \\
N &= \text{number of officers per week to allocate} = 25
\end{aligned}
$$