# Lesson 31. Review

**Problem 1.** Suppose that an optimization model with decision variables $w_1$ and $w_2$ has constraints

$$\begin{aligned} -2w_1 + w_2 &\leq 5 \\ w_1 &\geq 0. \end{aligned}$$

For each of the following tasks, show the correctness of your answer by solving the resulting model graphically.

a. Give a minimizing linear objective function for which the model has a unique optimal solution.
b. Give a minimizing linear objective function for which the model has multiple optimal solutions.
c. Give a minimizing linear objective function for which the model is unbounded.

**Problem 2.** Consider the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 10x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 + 4x_3 + 21x_4 = 13 \\ & 2x_1 + 6x_2 - 2x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Let the current basis be $\{x_1, x_3\}$.

a. Compute the current BFS.
b. Compute all the simplex directions at the current BFS.
c. Determine whether each of the simplex directions is improving.
d. Choose the "most improving" simplex direction and determine the maximum step size that preserves feasibility in that direction, and the new basis that would result after such a step.

**Problem 3.** Consider the following LP:

$$\begin{aligned} [\text{P}] \quad \max \quad & 3x_1 + 4x_2 + x_3 + 5x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 + 2x_4 \leq 5 \\ & 2x_1 + 3x_2 + x_3 + 3x_4 \leq 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

a. Write the dual of [P].
b. Graphically determine an optimal solution of the dual of [P].
c. Use complementary slackness to determine which two decision variables of [P] are equal to 0 in an optimal solution. Use this information to write an LP that determines the optimal values of the remaining two decision variables of [P].
d. Graphically determine an optimal solution for the LP in part c. Write an optimal solution for [P].

**Problem 4.** Gomoryco processes crude oil into aviation fuel and heating oil. It costs $c$ dollars to purchase 1000 barrels of crude oil, which is then distilled and yields 750 barrels of aviation fuel and 250 barrels of heating oil. Output from the distillation may be sold directly or processed in the catalytic cracker. If sold after distillation without further processing, aviation fuel sells for $p_a$ dollars per 1000 barrels, and heating oil sells for $p_h$ dollars per 1000 barrels. It takes $t_a$ hours to process 1000 barrels of aviation fuel in the catalytic cracker, and these 1000 barrels can be sold for $q_a$. It takes $t_h$ minutes to process 1000 barrels of heating oil in the cracker, and these 1000 barrels can be sold for $q_h$. Each day, at most $B$ barrels of crude oil can be purchased, and $T$ hours of cracker time are available. Formulate a linear program that maximizes Gomoryco's profits.

**Problem 5.** Suppose you are applying the simplex method to a canonical form LP with objective

$$\text{minimize} \quad 3w_1 + 11w_2 - 8w_3$$

Determine whether each of the following simplex directions for $w_4$ leads to a conclusion that the given LP is unbounded. Why?

a. $\mathbf{d}^{w_4} = (1, 0, -4, 1)$
b. $\mathbf{d}^{w_4} = (1, 3, 0, 1)$
c. $\mathbf{d}^{w_4} = (1, 0, 3, 1)$
d. $\mathbf{d}^{w_4} = (-1, 1, -2, 1)$

**Problem 6.** Santa Claus has a set of presents $P$ that he wants to distribute to a set of children $C$. Let $v_{ij}$ be the happiness value that child $i$ has for present $j$, for all $i \in C$ and $j \in P$. In addition, let $b_j$ be the number of present $j$ that Santa has available. Santa's objective is to distribute presents in a way that maximizes the happiness of the least lucky child. Formulate a linear program that helps Santa meet his objective. Assume that presents can be distributed fractionally.

*Note.* The "Santa Claus problem" is actually studied in the operations research and computer science literature as a fundamental resource allocation problem. It has particular importance in scheduling applications. See `http://dx.doi.org/10.1145/1132516.1132522` for an example.