Name:

SA305 – Linear Programming
Asst. Prof. Nelson Uhan

Spring 2015

Exam 2

Instructions

- You have 50 minutes to complete this exam.
- There are 3 problems on this exam, worth a total of 100 points.
- You may consult one 8.5 in × 11 in sheet of handwritten notes.
- You may use a calculator.
- **Show all your work.** Your answers should be legible and clearly labeled. It is your responsibility to make sure that I understand what you are doing. You will be awarded partial credit if your work merits it.
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until the end of 6th period on Monday 6 April.**

Problem 1. (50 total points)

Consider the following linear program.

$$
\begin{array}{llr}
\text{maximize} & 3x_1 + 2x_2 + x_3 & \\
\text{subject to} & 2x_1 - x_2 + x_3 = 6 & \text{(a)} \\
& 2x_1 + x_2 = 10 & \text{(b)} \\
& x_1 \geq 0 & \text{(c)} \\
& x_2 \geq 0 & \text{(d)} \\
& x_3 \geq 0 & \text{(e)} \\
& x_1 \leq 4 & \text{(f)} \\
& x_2 \leq 3 & \text{(g)} \\
& x_3 \leq 7 & \text{(h)}
\end{array}
$$

a. (30 points) For each of the following solutions, indicate whether they are (1) feasible, (2) basic solutions, and (3) extreme points. Explain why.

$$
\mathbf{x}^1 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \quad \mathbf{x}^2 = \begin{pmatrix} 0 \\ 10 \\ 16 \end{pmatrix} \quad \mathbf{x}^3 = \begin{pmatrix} 3.5 \\ 3 \\ 2 \end{pmatrix} \quad \mathbf{x}^4 = \begin{pmatrix} 3.75 \\ 2.5 \\ 1 \end{pmatrix}
$$

b. (18 points) Consider the solution $\mathbf{x}^1$ defined in part a. For each of the following directions, indicate whether they are (1) feasible at $\mathbf{x}^1$ and (2) improving at $\mathbf{x}^1$. Explain why.

$$\mathbf{d}^1 = \begin{pmatrix} -1/4 \\ 1/2 \\ 1 \end{pmatrix} \quad \mathbf{d}^2 = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} \quad \mathbf{d}^3 = \begin{pmatrix} -1/2 \\ 1 \\ 2 \end{pmatrix}$$

c. (2 points) Note this linear program is of the following form:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{x} \leq \mathbf{u} \end{aligned} \tag{$*$}$$

This is canonical form <u>except</u> that the decision variables also have simple upper bounds. Recall that:

In a canonical form linear program, any basic solution has m basic variables such that

(a) the columns of A corresponding to these m variables are linearly independent;
(b) the other $n - m$ variables are equal to 0.

Using the general definition of a basic solution, write a similar statement for linear programs of the form $(*)$.

Problem 2. (36 total points = 12 points × 3 parts)

Consider the following linear program:

$$
\begin{array}{lrrrrl}
\text{maximize} & 3x_1 + & 2x_2 - & x_3 & + 5x_4 & \\
\text{subject to} & x_1 & - 3x_2 + & x_3 & & = 7 \\
& & x_2 & + 2x_3 & + 4x_4 & = 10 \\
& x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

Professor I. M. Wright is performing the simplex method on this linear program. The current solution is $\mathbf{x}^\top = (7, 0, 0, 10)$. In each of the steps below, he has made a mistake. Find and correct his mistakes.

a. The simplex directions are $\mathbf{d}^{x_1} = (1, 0, -1, 1/2)$ and $\mathbf{d}^{x_2} = (3, 1, 0, -1/4)$.

b. The simplex direction $\mathbf{d}^{x_2}$ is improving because $\begin{pmatrix} 3 \\ 2 \\ -1 \\ 5 \end{pmatrix}^\top \begin{pmatrix} 7 \\ 0 \\ 0 \\ 10 \end{pmatrix} = 81.$

c. After Professor Wright chooses x_2 as the entering variable, he finds the step size to be $\lambda_{\max} = 7/3$.

Problem 3. (14 total points)

a. (4 points) Suppose $\mathbf{x}$ is a feasible solution to a linear program but is not a basic feasible solution. There are no improving, feasible directions at $\mathbf{x}$. Is $\mathbf{x}$ optimal? Explain why.

b. (5 points) A hat manufacturing plant produces baseball hats for colleges and professional leagues. Let H denote the set of all teams they make hats for, both college and professional. For $i \in H$, let the parameter

$$t_i = \begin{cases} 1 & \text{if hat } i \text{ is for a college team,} \\ 0 & \text{otherwise.} \end{cases}$$

Also for $i \in H$, let x_i denote a decision variable representing the number of type i hats made. Write a constraint that could be used in a linear program that would ensure that college team hats are at least 40% of the hats made. Use only the set, parameters, and decision variables given.

c. (5 points) Let $S = \{1, 2, 3, 4, 5\}$ and

$$Z_i = \{j \in S : j \leq i\} \quad \text{for } i \in S.$$

Define decision variables y_i for $i \in S$ and parameters a_i for $i \in Z_2$. Write the following constraints without for statements and summations:

$$\sum_{j \in S} y_j \geq a_i \text{ for } i \in Z_2.$$

Additional page for answers or scratchwork