1. (40 points) RugsMaker, Inc. is a manufacturer of carpets and toupees. To make each requires wool, weaving time, and padding material. A carpet requires 8 kg of wool, 3 hours of weaving time, and 500 g of padding material. A toupee requires 1 kg of wool, 5 hours of weaving time and 100 g of padding material. Each week, RugsMaker has 300 kg of wool and 18000 g of padding material. Their weavers can provide 80 hours of time for making carpets and toupees. RugsMaker sells carpets for $100 and toupees for $120.

(a) (*25 points*) Ignore integrality constraints and formulate a linear program that determines how many carpets and toupees RugsMaker should make to maximize revenues. Be sure to define all variables and explain all constraints.

(b) (*10 points*) Suppose RugsMaker wishes to ensure that at least one third of the wool they have is used to make toupees. Write a constraint that enforces this production goal. If you use new variables, be sure to define and constrain them appropriately.

(c) (*5 points*) Suppose that RugsMaker can hire part-time weavers to increase their weaving time. Each part-time weaving hour costs \$40. Formulate this option by defining any needed new variable(s) and indicating what changes to your formulation is needed. You do NOT need to rewrite your model, but do need to indicate what is to be changed.

2. (30 points) RugsMaker has decided to expand into the lucrative clown wig business. Their wig hair will either be curly or straight and the wig hair color will either be red or purple. To make their wigs, they start off with undyed hair and dye it at a cost of \$5 per kilogram. Each kg of ordinary undyed hair that undergoes the coloring process produces .6 kg of red hair, and .3 kg of purple hair. In addition, 1 kg of red hair can be further dyed to produce .9 kg of purple hair at a cost of \$2 per kg. The hair produced is naturally curly, but can be straightened at a cost of \$3 per kg straightened (there is no loss of hair when it is straightened). Hair must be dyed before it is straightened. All hair produced is shipped to their warehouse to await being sold. The process is shown in Figure 1.

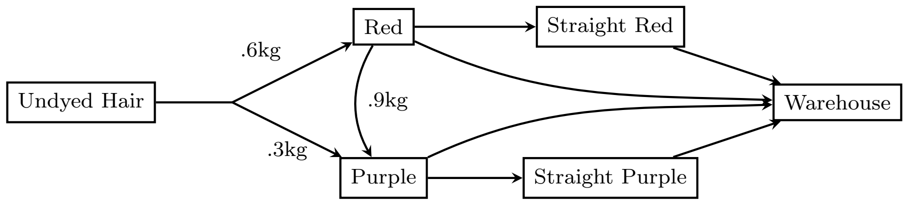

Figure 1: The clown wig production process. Conversions are 1 to 1 unless otherwise noted.

RugsMaker wants to ship at least 40 kg of curly red hair, at least 50 kg of curly purple hair, at least 60 kg of straight red hair, and at least 70 kg of straight purple hair to their warehouse.

Professor May B. Wright has begun this problem by writing the following variable definitions:

H = total kg of undyed hair used

R_S = kg of red hair straightened

R_W = kg of curly red hair shipped to warehouse

R_{SW} = kg of straight red hair shipped to warehouse

R_P = kg of curly red hair dyed purple

P_S = kg of purple hair straightened

P_W = kg of curly purple hair shipped to warehouse

P_{SW} = kg of straight purple hair shipped to warehouse

(a) (*10 points*) Using Prof. Wright's variables, write the objective for a linear program that determines the minimum cost way to meet the shipping requirements via the given clown wig production process.

(b) (*15 points*) Using Prof. Wright's variables, write the constraints for a linear program that determines the minimum cost cost way to meet the shipping requirements via the given clown wig production process.

(c) (*5 points*) Professor I.M. Wright states, "This problem must be unbounded! There are no constraints on the amount of undyed hair that is used!!" Is he correct or not? Justify your answer.

3. (30 points) Let $R = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0, x + y + z = 2, x \geq 0, y \geq 0, z \geq 0\}$.
(a) (*6 points*) What are the basic solutions of R? Justify your answer.

(b) (*6 points*) What are the extreme points of R? Justify your answer.

(c) (*4 points*) Consider the linear program $\max\{x - y : (x, y, z) \in R\}$, i.e.,

$$\begin{aligned} \max \quad & x - y \\ \text{s.t.} \quad & x + y - z = 0 \\ & x + y + z = 2 \\ & x \geq 0, y \geq 0, z \geq 0. \end{aligned} \tag{P}$$

Note that the feasible region is R! What is the optimal solution? Explain your answer.

(d) (*6 points*) At the point $(1, 0, 1)$, is $\mathbf{d} = (d_1, d_2, d_3) = (-1, 1, 0)$ a feasible direction for the linear program (P)? Justify your answer.

(e) (*6 points*) At the point $(1, 0, 1)$, is $\mathbf{d} = (d_1, d_2, d_3) = (-1, 1, 0)$ an improving direction for the linear program (P)? Justify your answer.

(f) (*2 points*) Consider the point $(1/2, 1/2, 1) \in R$. For $\theta \in [0, 2\pi)$, consider vectors of the form $\mathbf{d}(\theta) = (\cos(\theta), \sin(\theta), 0)$. For what values of θ is $\mathbf{d}(\theta)$ a feasible direction?

4. (35 points) Suppose you are in the midst of solving the following canonical form LP using the simplex method:

$$\begin{array}{lrcrcrcrcl} \text{maximize} & & & 7x_2 & & & + & 7x_4 & & \\ \text{subject to} & x_1 & + & 2x_2 & & & + & 10x_4 & & = 45 \\ & & & 2x_2 & + & x_3 & + & 2x_4 & & = 13 \\ & & & 2x_2 & & & + & x_4 & + x_5 & = 9 \\ & x_1, & & x_2, & & x_3, & & x_4, & x_5 & \geq 0 \end{array}$$

(a) (*6 points*) Suppose the current basis is $\{x_1, x_3, x_5\}$. What is the current basic feasible solution?

(b) (*12 points*) Compute all the simplex directions at the basic feasible solution you found in part (a).

(c) (*8 points*) Determine whether each of the simplex directions you found in part (b) are improving.

(d) (*6 points*) Use your answers in parts (b) and (c) to find another basic feasible solution with a better objective function value and its corresponding basis, or explain why this is impossible.

(e) (*3 points*) Suppose you continue solving the above LP using the simplex method until you reach the solution $(0, 5/2, 0, 4, 0)$ with basis $\{x_1, x_2, x_4\}$. You compute the following simplex directions and reduced costs (recall that the objective function is $f(\mathbf{x}) = 7x_2 + 7x_4$):

$$\mathbf{d}^{x_3} = \left(9, \frac{1}{2}, 1, -1, 0\right) \qquad \bar{c}_{x_3} = \nabla f^\top \mathbf{d}^{x_2} = -\frac{7}{2}$$
$$\mathbf{d}^{x_5} = (-8, -1, 0, 1, 1) \qquad \bar{c}_{x_5} = \nabla f^\top \mathbf{d}^{x_5} = 0$$

Professor I.M. Wright looks over your shoulder and declares:

> "The solution $(0, 5/2, 0, 4, 0)$ must be optimal because all the reduced costs are less than or equal to zero! In addition, there must be multiple optimal solutions, since you can move from $(0, 5/2, 0, 4, 0)$ in the direction $\mathbf{d}^{x_5}$ towards other solutions with the same objective function value!"

(This guy is pretty annoying.) Is he right? Why or why not?

5. (25 points) Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 - 4x_3 \\ \text{subject to} \quad & x_1 \leq 10 + x_2 + 5x_3 \\ & -3x_3 + 9x_1 = -6 \\ & x_1 \geq 0, x_3 \leq 0 \end{aligned}$$

(a) (*12 points*) Convert this LP into canonical form.

(b) (*8 points*) Write the Phase I LP for the canonical form LP you wrote in part (a).

(c) (*5 points*) Suppose the optimal value of the Phase I LP you wrote in part (b) is 6. What does this mean?

6. (35 points)

(a) (*10 points*) Consider the following parameters and variables:

For $t \in \mathcal{T} = \{3, 4, 5, 6\}$, let
x_t = decision variable for number of widgets produced in time t,
s_t = decision variable for number of widgets stored at the end of time t,
d_t = parameter for demand for widgets in time t.

Also, you are told that $s_2 = 50$ and that demand in a time period can be met by production in that period. Do the following inequalities correctly model the inventory/demand constraint? Briefly explain your answer.

$$s_{t-1} + x_t \geq s_t + d_t, \text{ for } t \in \mathcal{T}.$$

(b) (*10 points*) Consider the following MathProg/Gusek code written for a work scheduling problem:

```
set Days := M T W Th F Sa Su;

set SomeSet := M T Sa Su;

var w{d in Days} >= 0;  ##workers starting work on day d
```

You are told that workers work for 4 consecutive days and then have three days off. Briefly explain what the following constraint enforces:

```
subj to whatisthis: sum{d in SomeSet} w[d] >= 12;
```

(c) (*8 points*) Using a bounds argument, derive the dual for the following linear program:

$$\begin{aligned} \text{minimize} \quad & 2x_1 - 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 - 5x_3 \leq 10 \\ & -3x_1 + 9x_2 \geq -2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \leq 0 \end{aligned}$$

(d) (*5 points*) Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & 2x_1 - 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 + 5x_3 + s_1 = 10 \\ & -3x_1 + 9x_2 - s_2 = -2 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

You are told that an optimal basic feasible solution is $(0, 5, 0, 0, 47)$. What is the associated optimal dual solution?

(e) (*2 points*) Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 2 \\ & x_1 + x_2 = 1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The optimal primal basic feasible solution is $(x_1, x_2, x_3) = (0, 1, 0)$. Describe **all** dual optimal solutions.

Name (please print): ______________________________

Only write your name above!

Instructions:

- No books are allowed. **One** 8.5 by 11 inch formula/note sheet is allowed.
- Show all work clearly. (little or no credit will be given for a numerical answer without the correct accompanying work. Partial credit is given where appropriate.)
- If you need more space than is provided, use the back of the previous page.
- Please read the question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.

Problem	Points	Score
1	40	
2	30	
3	30	
4	35	
5	25	
6	35	
Total	195	