

**Problem.** Alvin Fine produces three perfumes from raw material. Thirty thousand ounces of raw material is available. Each ounce of raw material can be transformed into 0.4 ounces of perfume 1, 0.3 ounces of perfume 2, and 0.2 ounces of perfume 3, while 0.1 ounces is lost as waste material. Each ounce of perfume 1 can be further processed into 0.6 ounces of perfume 2, 0.3 ounces of perfume 3, and 0.1 ounces of waste material. Alvin Fine has been contracted to produce at least 4000 ounces of perfume 1, 8000 ounces of perfume 2, and 10000 ounces of perfume 3. Because of its environmental initiatives it wishes to minimize waste material. Formulate a linear program that determines how much perfume to produce while minimizing waste.

In class, we proposed one linear program for this problem. Here are two other equivalent models.

**Alternate solution 1.**

*Decision variables.*

- $r$  = amount of raw material used
- $x_{1r}$  = amount of raw material transformed into perfume 1
- $x_{2r}$  = amount of raw material transformed into perfume 2
- $x_{3r}$  = amount of raw material transformed into perfume 3
- $w_r$  = amount of raw material transformed into waste
- $x_{1f}$  = amount of perfume 1 further processed
- $x_{21}$  = amount of perfume 1 transformed into perfume 2
- $x_{31}$  = amount of perfume 1 transformed into perfume 3
- $w_1$  = amount of perfume 1 transformed into waste
- $x_1$  = amount of perfume 1 produced to meet demand
- $x_2$  = amount of perfume 2 produced to meet demand
- $x_3$  = amount of perfume 3 produced to meet demand

*Objective function and constraints.*

- minimize  $w_r + w_1$  (1a)
- subject to  $r \leq 30000$  (1b)
- $x_{1r} = 0.4r, x_{2r} = 0.3r, x_{3r} = 0.2r, w_r = 0.1r$  (1c)
- $x_{1r} = x_{1f} + x_1$  (1d)
- $x_{21} = 0.6x_{1f}, x_{31} = 0.3x_{1f}, w_1 = 0.1x_{1f}$  (1e)
- $x_2 = x_{2r} + x_{21}$  (1f)
- $x_3 = x_{3r} + x_{31}$  (1g)
- $x_1 \geq 4000, x_2 \geq 8000, x_3 \geq 10000$  (1h)
- $r, x_{1r}, x_{2r}, x_{3r}, w_r, x_{1f}, x_{21}, x_{31}, w_1, x_1, x_2, x_3 \geq 0$  (1i)

The objective (1a) minimizes the total waste produced. Constraint (1b) ensures that Alvin Fine does not use more raw material than is available. Constraints (1c) ensures that raw material is transformed into the three perfumes and waste in the correct proportions. Constraint (1d) ensures that the amount of perfume 1 transformed from raw material,  $x_{1r}$ , is equal to the amount of perfume 1 produced to meet demand,  $x_1$ , plus the amount of perfume 1 further processed,  $x_{1f}$ . Constraints (1e) ensures that the perfume 1 that is further processed is transformed into the two other perfumes and waste in the correct proportions. Constraint (1f) ensures

that the amount of perfume 2 produced to meet demand,  $x_2$ , is equal to the amount of perfume 2 produced from both raw material,  $x_{2r}$ , and perfume 1,  $x_{21}$ . Constraint (1g) does the same as constraint (1f), except for perfume 3. Constraints (1h) ensure that demands for the three perfumes are met. Finally, constraints (1i) ensure that the amounts of perfumes and waste are nonnegative.

### Alternate solution 2.

*Decision variables.*

- $r$  = amount of raw material used
- $y$  = amount of perfume 1 further processed
- $x_1$  = amount of perfume 1 produced to meet demand
- $x_2$  = amount of perfume 2 produced to meet demand
- $x_3$  = amount of perfume 3 produced to meet demand

*Objective function and constraints.*

$$\begin{array}{ll} \text{minimize} & 0.1r + 0.1y & (2a) \\ \text{subject to} & r \leq 30000 & (2b) \\ & y + x_1 = 0.4r & (2c) \\ & x_2 = 0.3r + 0.6y & (2d) \\ & x_3 = 0.2r + 0.3y & (2e) \\ & x_1 \geq 4000, x_2 \geq 8000, x_3 \geq 10000 & (2f) \\ & r, y, x_1, x_2, x_3 \geq 0 & (2g) \end{array}$$

The objective (2a) minimizes the total waste produced. Constraint (2b) ensures that Alvin Fine does not use more raw material than is available. Constraint (2c) ensures that the amount of perfume 1 transformed from raw material,  $0.4r$ , is equal to the amount of perfume 1 produced to meet demand,  $x_1$ , plus the amount of perfume 1 further processed,  $y$ . Constraint (2d) ensures that the amount of perfume 2 produced to meet demand,  $x_2$ , is equal to the amount of perfume 2 produced from both raw material,  $0.3r$ , and perfume 1,  $0.6y$ . Constraint (2e) does the same as constraint (2d), except for perfume 3. Constraints (2f) ensure that demands for the three perfumes are met. Finally, constraints (2g) ensure that the amounts of perfumes are nonnegative.

**Commentary.** It's not too hard to see that these two linear programs are equivalent (this is left as an exercise for you). Although the first model has more variables, one might argue that it is easier to understand, since the decision variables and constraints more closely reflect the different stages of Alvin Fine's production process. Of course, this is somewhat a matter of personal taste.