SA305 – Linear Programming   Spring 2016
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Lesson 16. Improving Search: Convexity and Optimality

1 Overview

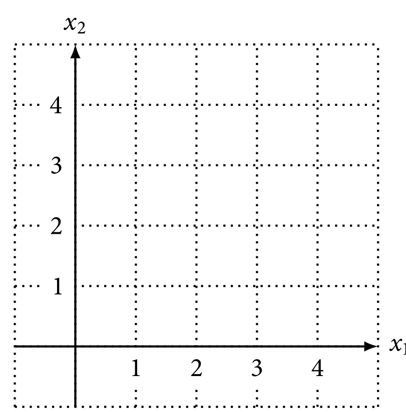
1: Find an initial feasible solution $\mathbf{x}^0$
2: Set $k = 0$
3: **while** $\mathbf{x}^k$ is not locally optimal **do**
4: Determine a new feasible solution $\mathbf{x}^{k+1}$ that improves the objective value at $\mathbf{x}^k$
5: Set $k = k + 1$
6: **end while**

- Step 3 – Improving search converges to local optimal solutions, which aren't necessarily globally optimal
- Wishful thinking: when are all local optimal solutions are in fact globally optimal?

2 Convex sets

Example 1. Let $\mathbf{x} = (1,1)$ and $\mathbf{y} = (4,3)$. Compute and plot $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ for $\lambda \in \{0, 1/3, 2/3, 1\}$.

λ	$\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$
0	
1/3	
2/3	
1	



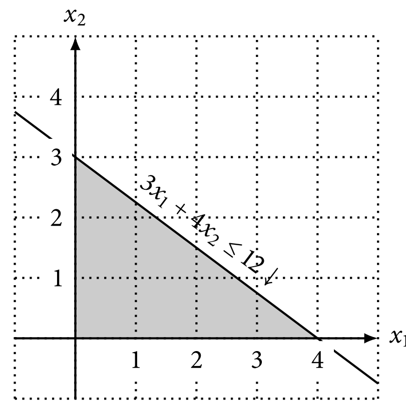
- Given two solutions $\mathbf{x}$ and $\mathbf{y}$, the **line segment** joining them is

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \quad \text{for } \lambda \in [0,1]$$

- A feasible region S is **convex** if for all $\mathbf{x}, \mathbf{y} \in S$, then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$ for all $\lambda \in [0,1]$
  - A feasible region is convex if for any two solutions in the region, all solutions on the line segment joining these solutions are also in the region
- Graphically: convex vs. nonconvex

Example 2. Show that the feasible region of the LP below is convex.

$$\begin{aligned} \text{minimize} \quad & 3x_1 + x_2 & \\ \text{subject to} \quad & 3x_1 + 4x_2 \leq 12 & (1) \\ & x_1 \geq 0 & (2) \\ & x_2 \geq 0 & (3) \end{aligned}$$



Proof.

- Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be arbitrary points in the feasible region
- In other words, $\mathbf{x}$ and $\mathbf{y}$ satisfy (1), (2), (3)
- We need to show $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ also satisfies (1), (2), (3) for any $\lambda \in [0, 1]$
- Note that

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} = \boxed{}$$

- One constraint at a time: does $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ satisfy (1)?

$$\begin{aligned} 3\big(\lambda x_1 + (1 - \lambda) y_1\big) + 4\big(\lambda x_2 + (1 - \lambda) y_2\big) &= \lambda\big(3x_1 + 4x_2\big) + (1 - \lambda)\big(3y_1 + 4y_2\big) \\ &\leq 12\lambda + 12(1 - \lambda) \\ &= 12 \end{aligned}$$

- We can show $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ also satisfies (2) and (3) in a similar fashion □

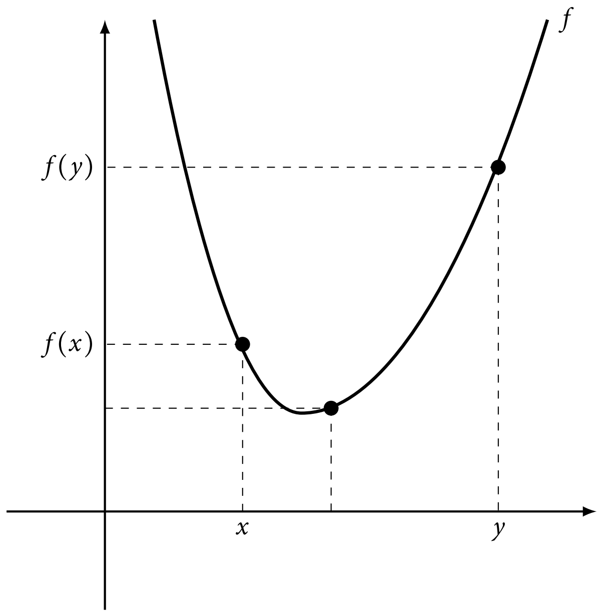
- **In general, the feasible region of an LP is convex**

3 Convex functions

- Given a convex feasible region S, a function $f(\mathbf{x})$ is **convex** if for all solutions $\mathbf{x}, \mathbf{y} \in S$ and for all $\lambda \in [0,1]$

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$$

- Graphically:



Example 3. Show that the objective function of the LP in Example 2 is convex.

Proof.

- Let $f(\mathbf{x}) = 3x_1 + x_2$
- For any $\mathbf{x}$ and $\mathbf{y}$, we have:

$$\begin{aligned} f\big(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\big) &= 3\big(\lambda x_1 + (1-\lambda) y_1\big) + \big(\lambda x_2 + (1-\lambda) y_2\big) \\ &= \lambda\big(3x_1 + x_2\big) + (1-\lambda)\big(3y_1 + y_2\big) \\ \\ &= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \qquad \square \end{aligned}$$

- **In general, the objective function of an LP – a linear function – is convex**

4 Minimizing convex functions over convex sets

Big Theorem. Consider the following optimization model:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} b_i \quad \text{for } i \in \{1, \dots, m\} \end{aligned} \tag{$*$}$$

Suppose f is convex and the feasible region is convex. If $\mathbf{x}$ is a local optimal solution, then $\mathbf{x}$ is a global optimal solution.

Proof.

- By contradiction – suppose $\mathbf{x}$ is not a global optimal solution
- Then there must be another feasible solution $\mathbf{y}$ such that $f(\mathbf{y}) < f(\mathbf{x})$
- Take $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ really close to $\mathbf{x}$ (λ really close to 1)
- Since the feasible region is convex, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also in the feasible region
- We have that:

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &\leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) && \text{(since } f \text{ is convex)} \\ &< \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{x}) && \text{(since } f(\mathbf{y}) < f(\mathbf{x})\text{)} \\ &= f(\mathbf{x}) \end{aligned}$$

- Therefore: $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < f(\mathbf{x})$
- $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is a feasible solution in the neighborhood of $\mathbf{x}$ with better objective value than $\mathbf{x}$
- This contradicts $\mathbf{x}$ being a local optimal solution!
- Therefore, $\mathbf{x}$ must be a global optimal solution □

- Remember that an improving search algorithm finds local optimal solutions
- Since the objective function of an LP is convex, and the feasible region of an LP is convex:

Big Corollary 1. A global optimal solution of a minimizing linear program can be found with an improving search algorithm.

- A similar theorem and corollary exists when maximizing concave functions over convex sets
  - See pages 222–225 in Rader for details

Big Corollary 2. A global optimal solution of a maximizing linear program can be found with an improving search algorithm.