

Lesson 18. Linear Programs in Canonical Form

0 Warm up

Example 1.

Let $A = \begin{pmatrix} 1 & 9 & 8 \\ 5 & 2 & 3 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then $A\mathbf{x} =$

1 Canonical form

- LP in **canonical form** with decision variables x_1, \dots, x_n :

$$\begin{aligned} &\text{minimize / maximize} && \sum_{j=1}^n c_j x_j \\ &\text{subject to} && \sum_{j=1}^n a_{ij} x_j = b_i \quad \text{for } i \in \{1, \dots, m\} \\ &&& x_j \geq 0 \quad \text{for } j \in \{1, \dots, n\} \end{aligned}$$

- In vector-matrix notation with decision variable vector $\mathbf{x} = (x_1, \dots, x_n)$:

$$\begin{aligned} &\text{minimize / maximize} && \mathbf{c}^T \mathbf{x} \\ &\text{subject to} && A\mathbf{x} = \mathbf{b} \\ &&& \mathbf{x} \geq \mathbf{0} \end{aligned}$$

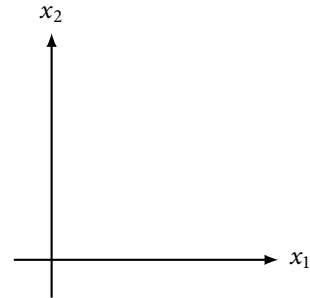
- A has m rows and n columns, \mathbf{b} has m components, and \mathbf{c} and \mathbf{x} each have n components

- We typically assume that $m \leq n$, and $\text{rank}(A) = m$

Example 2. Identify \mathbf{x} , \mathbf{c} , A , and \mathbf{b} in the following canonical form LP:

$$\begin{aligned} &\text{maximize} && 3x + 4y - z \\ &\text{subject to} && 2x - 3y + z = 10 \\ &&& 7x + 2y - 8z = 5 \\ &&& x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

- A canonical form LP always has at least 1 extreme point (if it has a feasible solution)
 - Intuition: if solutions in the feasible region must satisfy $\mathbf{x} \geq \mathbf{0}$, then the feasible region must be “pointed”



2 Converting any LP to an equivalent canonical form LP

- Inequalities \rightarrow equalities
 - **Slack** and **surplus** variables “consume the difference” between the LHS and RHS
 - If constraint i is a \leq -constraint, add a slack variable s_i :

$$\sum_{j=1}^n a_{ij}x_j \leq b_i \quad \Rightarrow$$

- If constraint i is a \geq -constraint, subtract a surplus variable s_i :

$$\sum_{j=1}^n a_{ij}x_j \geq b_i \quad \Rightarrow$$

- Nonpositive variables \rightarrow nonnegative variables
 - If $x_j \leq 0$, then introduce a new variable x'_j and substitute $x_j = -x'_j$ everywhere – in particular:

- Unrestricted (“free”) variables \rightarrow nonnegative variables
 - If x_j is unrestricted in sign, introduce 2 new nonnegative variables x_j^+, x_j^-
 - Substitute $x_j = x_j^+ - x_j^-$ everywhere
 - Why does this work?
 - ◊ Any real number can be expressed as the difference of two nonnegative numbers

Example 3. Convert the following LPs to canonical form.

$$\begin{aligned} &\text{maximize} && 3x + 8y \\ &\text{subject to} && x + 4y \leq 20 \\ &&& x + y \geq 9 \\ &&& x \geq 0, y \text{ free} \end{aligned}$$

$$\begin{aligned} &\text{minimize} && 5x_1 - 2x_2 + 9x_3 \\ &\text{subject to} && 3x_1 + x_2 + 4x_3 = 8 \\ &&& 2x_1 + 7x_2 - 6x_3 \leq 4 \\ &&& x_1 \leq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

