SA305 – Linear Programming | Spring 2016
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Lesson 19. Basic Solutions in Canonical Form LPs

1 Overview

- Recall: an LP in **canonical form** looks like

$$\begin{aligned} \text{minimize / maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{CF}$$

  - All decision variables are nonnegative
  - All general constraints are equalities

- Also recall: a solution $\mathbf{x}$ of an LP with n decision variables is a **basic solution** if
  (a) it satisfies all equality constraints
  (b) at least n constraints are active at $\mathbf{x}$ and are linearly independent

- The solution $\mathbf{x}$ is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of the LP

- This lesson: what do basic solutions in canonical form LPs look like?

2 Example

- Consider the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 3x + 8y \\ \text{subject to} \quad & x + 4y + s_1 = 20 && (1) \\ & x + y + s_2 = 9 && (2) \\ & 2x + 3y + s_3 = 20 && (3) \\ & x \geq 0 && (4) \\ & y \geq 0 && (5) \\ & s_1 \geq 0 && (6) \\ & s_2 \geq 0 && (7) \\ & s_3 \geq 0 && (8) \end{aligned}$$

- Identify the matrix A and the vectors $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{b}$ in the above canonical form LP.

- Suppose $\mathbf{x}$ is a basic solution
  - How many linearly independent constraints must be active at $\mathbf{x}$?
  - How many of these must be equality constraints?
  - How many of these must be nonnegativity bounds?
- Let's compute the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8)
  - It turns out that the constraints (1), (2), (3), (6), and (8) are linearly independent
  - Since the basic solution is active at the nonnegativity bounds (6) and (8),
  - The other variables, x, y, and s_2 are potentially nonzero
  - Substituting $s_1 = 0$ and $s_3 = 0$ into the other constraints (1), (2), and (3), we get

$$\begin{aligned} x + 4y + (0) \qquad\qquad &= 20 \\ x + y \qquad + s_2 \qquad &= 9 \\ 2x + 3y \qquad\qquad + (0) &= 20 \end{aligned} \tag{$*$}$$

  - Let $\mathbf{x}_B = (x, y, s_2)$ and B be the submatrix of A consisting of columns corresponding to x, y, and s_2:

$$B = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

  - Note that $(*)$ can be written as

$$B\mathbf{x}_B = \mathbf{b} \tag{$**$}$$

  - The columns of B linearly independent. Why?
  - $(**)$ has a unique solution. Why?

- It turns out that the solution to $(**)$ is $\mathbf{x}_B = (4, 4, 1)$
- Put it together: the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8) is

3 Generalizing the example

- Now let's generalize what happened in the example above
- Consider the generic canonical form LP (CF)
  - Let n = number of decision variables
  - Let m = number of equality constraints
  - In other words, A has m rows and n columns
  - Assume $m \leq n$ and $\text{rank}(A) = m$
- Suppose $\mathbf{x}$ is a basic solution
  - How many linearly independent constraints must be active at $\mathbf{x}$?
  - Since $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, how many nonnegativity bounds must be active?
- Generalizing our observations from the example, we have the following theorem:

Theorem 1. If $\mathbf{x}$ is a basic solution of a canonical form LP, then there exists m **basic variables** of $\mathbf{x}$ such that

(a) the columns of A corresponding to these m variables are linearly independent;
(b) the other $n - m$ **nonbasic variables** are equal to 0.

The set of basic variables is referred to as the **basis** of $\mathbf{x}$.

- Let's check our understanding of this theorem with the example
  - Back in the example, n = and m =
  - Recall that $\mathbf{x} = (x, y, s_1, s_2, s_3) = (4, 4, 0, 1, 0)$ is a basic solution
  - Which variables of $\mathbf{x}$ correspond to m LI columns of A?
  - Which $n - m$ variables of $\mathbf{x}$ are equal to 0?

- The basic variables of $\mathbf{x}$ are
- The nonbasic variables of $\mathbf{x}$ are
- The basis of $\mathbf{x}$ is

- Let B be the submatrix of A consisting of columns corresponding to the m basic variables
- Let $\mathbf{x}_B$ be the vector of these m basic variables
- Since the columns of B are linearly independent, the system $B\mathbf{x}_B = \mathbf{b}$ has a unique solution
  - This matches what we saw in $(**)$ in the above example
- The m basic variables are potentially nonzero, while the other $n - m$ nonbasic variables are forced to be zero