SA305 – Linear Programming
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Lesson 21. The Simplex Method – Example

Problem 1. Consider the following LP

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 \leq 18 \\ & 4x_1 + 2x_2 + 5x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{1}$$

The canonical form of this LP is

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 + s_1 = 18 \\ & 4x_1 + 2x_2 + 5x_3 + s_2 = 10 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned} \tag{2}$$

a. Use the simplex method to solve the canonical form LP (2). In particular:
   - Use the initial BFS $\mathbf{x}^0 = (0, 0, 0, 18, 10)$ with basis $\mathcal{B}^0 = \{s_1, s_2\}$.
   - Choose your entering variable using **Dantzig's rule** – that is, choose the improving simplex direction with the most positive reduced cost. (If this was a minimization LP, you would choose the improving simplex direction with the most negative reduced cost.)

b. What is the optimal value of the canonical form LP (2)? Give an optimal solution.

c. What is the optimal value of the original LP (1)? Give an optimal solution.

a+b. $\vec{x}^0 = (0,0,0,18,10) \qquad \mathcal{B}^0 = \{s_1, s_2\}$

$\underline{\vec{d}^{x_1}}$: $\vec{d}^{x_1} = (1,0,0,d_{s_1},d_{s_2})$

$A\vec{d}^{x_1} = 0$: $\quad 2 + d_{s_1} = 0$
$\quad 4 + d_{s_2} = 0$

$\Rightarrow \vec{d}^{x_1} = (1,0,0,-2,-4)$

$\bar{c}_{x_1} = 4$

$\underline{\vec{d}^{x_2}}$: $\vec{d}^{x_2} = (0,1,0,d_{s_1},d_{s_2})$

$A\vec{d}^{x_2} = 0$: $\quad -1 + d_{s_1} = 0$
$\quad 2 + d_{s_2} = 0$

$\Rightarrow \vec{d}^{x_2} = (0,1,0,1,-2)$

$\bar{c}_{x_2} = 3$

$\underline{\vec{d}^{x_3}}$: $\vec{d}^{x_3} = (0,0,1,d_{s_1},d_{s_2})$

$A\vec{d}^{x_3} = 0$: $\quad 4 + d_{s_1} = 0$
$\quad 5 + d_{s_2} = 0$

$\Rightarrow \vec{d}^{x_3} = (0,0,1,-4,-5)$

$\boxed{\bar{c}_{x_3} = 5}$ choose x_3 as entering

$\underline{\text{MRT}}$: $\lambda_{max} = \min\left\{ \overset{s_1}{\frac{18}{4}}, \overset{s_2}{\frac{10}{5}} \right\} = 2 \qquad s_2$ is leaving

$\Rightarrow \vec{x}^1 = \vec{x}^0 + \lambda_{max}\vec{d}^{x_3} = (0,0,2,10,0) \qquad \mathcal{B}^1 = \{x_3, s_1\}$

$\vec{x}^1 = (0, 0, 2, 10, 0) \qquad \mathcal{B}^1 = \{x_3, s_1\}$

$\underline{\vec{d}^{x_1}}$: $\vec{d}^{x_1} = (1, 0, d_{x_3}, d_{s_1}, 0)$

$A\vec{d}^{x_1} = 0: \quad \begin{array}{l} 2 + 4d_{x_3} + d_{s_1} = 0 \\ 4 + 5d_{x_3} = 0 \end{array}$

$\Rightarrow \vec{d}^{x_1} = (1, 0, -\frac{4}{5}, \frac{6}{5}, 0)$

$\bar{c}_{x_1} = 0$

$\underline{\vec{d}^{x_2}}$: $\vec{d}^{x_2} = (0, 1, d_{x_3}, d_{s_1}, 0)$

$A\vec{d}^{x_2} = 0: \quad \begin{array}{l} -1 + 4d_{x_3} + d_{s_1} = 0 \\ 2 + 5d_{x_3} = 0 \end{array}$

$\Rightarrow \vec{d}^{x_2} = (0, 1, -\frac{2}{5}, \frac{13}{5}, 0)$

$\boxed{\bar{c}_{x_2} = 1}$ choose x_2 as entering

$\underline{\vec{d}^{s_2}}$: $\vec{d}^{s_2} = (0, 0, d_{x_3}, d_{s_1}, 1)$

$A\vec{d}^{s_2} = 0: \quad \begin{array}{l} 4d_{x_3} + d_{s_1} = 0 \\ 1 + 5d_{x_3} = 0 \end{array}$

$\Rightarrow \vec{d}^{x_2} = (0, 0, -\frac{1}{5}, \frac{4}{5}, 1)$

$\bar{c}_{s_2} = -1$

$\underline{\text{MRT}}$: $\lambda_{max} = \min\left\{ \overset{x_3}{\frac{2}{2/5}} \right\} = 5$ $\quad$ x_3 is leaving

$\Rightarrow \vec{x}^2 = \vec{x}^1 + \lambda_{max}\vec{d}^{x_2} = (0, 5, 0, 23, 0) \qquad \mathcal{B}^2 = \{x_2, s_1\}$

$\vec{x}^2 = (0, 5, 0, 23, 0) \qquad \mathcal{B}^2 = \{x_2, s_1\}$

$\underline{\vec{d}^{x_1}}$: $\vec{d}^{x_1} = (1, d_{x_2}, 0, d_{s_1}, 0)$

$A\vec{d}^{x_1} = 0: \quad \begin{array}{l} 2 - d_{x_2} + d_{s_1} = 0 \\ 4 + 2d_{x_2} = 0 \end{array}$

$\Rightarrow \vec{d}^{x_1} = (1, -2, 0, -4, 0)$

$\bar{c}_{x_1} = -2$

$\underline{\vec{d}^{x_3}}$: $\vec{d}^{x_3} = (0, d_{x_2}, 1, d_{s_1}, 0)$

$A\vec{d}^{x_3} = 0: \quad \begin{array}{l} 4 - d_{x_2} + d_{s_1} = 0 \\ 5 + 2d_{x_2} = 0 \end{array}$

$\Rightarrow \vec{d}^{x_3} = (0, -\frac{5}{2}, 1, -\frac{13}{2}, 0)$

$\bar{c}_{x_3} = -\frac{5}{2}$

$\underline{\vec{d}^{s_2}}$: $\vec{d}^{s_2} = (0, d_{x_2}, 0, d_{s_1}, 1)$

$A\vec{d}^{s_2} = 0: \quad \begin{array}{l} -d_{x_2} + d_{s_1} = 0 \\ 1 + 2d_{x_2} = 0 \end{array}$

$\Rightarrow \vec{d}^{s_2} = (0, -\frac{1}{2}, 0, -\frac{1}{2}, 1)$

$\bar{c}_{s_2} = -\frac{3}{2}$

No simplex directions are improving $\Rightarrow$ $\underline{\vec{x}^2 \text{ is optimal with value } 15}$

c. In the original LP, $x_1 = 0$, $x_2 = 5$, $x_3 = 0$ is an optimal solution with value 15