Lesson 23. Degeneracy, Convergence, Multiple Optimal Solutions

0 Warm up

Example 1. Suppose we are using the simplex method to solve the following canonical form LP:

$$
\begin{aligned}
\text{maximize} \quad & 10x + 3y \\
\text{subject to} \quad & x + y + s_1 = 4 && (1)\\
& 5x + 2y + s_2 = 11 && (2)\\
& y + s_3 = 4 && (3)\\
& x \geq 0 && (4)\\
& y \geq 0 && (5)\\
& s_1 \geq 0 && (6)\\
& s_2 \geq 0 && (7)\\
& s_3 \geq 0 && (8)
\end{aligned}
$$

Let $\mathbf{x} = (x, y, s_1, s_2, s_3)$. Our current BFS is $\mathbf{x}^t = (0, 4, 0, 3, 0)$ with basis $\mathcal{B}^t = \{y, s_1, s_2\}$. The simplex directions are $\mathbf{d}^x = (1, 0, -1, -5, 0)$ and $\mathbf{d}^{s_3} = (0, -1, 1, 2, 1)$. Compute $\mathbf{x}^{t+1}$ and $\mathcal{B}^{t+1}$.

- In the above example, the step size $\lambda_{\max} = 0$
- As a result, $\mathbf{x}^{t+1} = \mathbf{x}^t$: it looks like our solution didn't change!
- The basis did change, however: $\mathcal{B}^{t+1} \neq \mathcal{B}^t$
- Why did this happen?

1 Degeneracy

- A BFS $\mathbf{x}$ of an LP with n decision variables is **degenerate** if there are more than n constraints active at $\mathbf{x}$
  - i.e. there are multiple collections of n linearly independent constraints that define the same $\mathbf{x}$

Example 2. Is $\mathbf{x}^t$ in Example 1 degenerate? Why?

- In $\mathbf{x}^t = (0, 4, 0, 3, 0)$ in Example 1, "too many" of the nonnegativity constraints are active
  - As a result, some of the basic variables are equal to zero
- Recall: a BFS of a canonical form LP with n decision variables and m equality constraints has
  - ______ basic variables, potentially zero or nonzero
  - ______ nonbasic variables, always equal to 0
- Suppose $\mathbf{x}$ is a degenerate BFS, with $n + k$ active constraints ($k \geq 1$)
- Then ______ nonnegativity bounds must be active, which is larger than $n - m$
- Therefore: a BFS $\mathbf{x}$ of a canonical form LP is degenerate if

- As a result, a degenerate BFS may correspond to several bases
  - e.g. in Example 1, the BFS $(0, 4, 0, 3, 0)$ has bases: ______
- Every step of the simplex method
  - does not necessarily move to a geometrically adjacent extreme point
  - does move to an adjacent BFS (in particular, the bases differ by exactly 1 variable)
- At a degenerate BFS, the simplex method might "get stuck" for a few steps
  - Same BFS, different bases, different simplex directions
  - Zero-length moves: $\lambda_{\max} = 0$
- When $\lambda_{\max} = 0$, just proceed as usual
- Simplex computations will normally escape a sequence of zero-length moves and move away from the current BFS

2 Convergence

- In extreme cases, degeneracy can cause the simplex method to cycle over a set of bases that all represent the same extreme point
  - See Rader p. 291 for an example
- Can we guarantee that the simplex method terminates?
- Yes! Anticycling rules exist
- Easy anticycling rule: **Bland's rule**
  - Fix an ordering of the decision variables and rename them so that they have a common index
    - e.g. $(x, y, s_1, s_2, s_3) \rightarrow (x_1, x_2, x_3, x_4, x_5)$
  - Entering variable: choose nonbasic variable with smallest index among those corresponding to improving simplex directions
  - Leaving variable: choose basic variable with smallest index among those that define $\lambda_{\max}$

3 Multiple optimal solutions

- Suppose our current BFS is $\mathbf{x}^t$, and y is the entering variable
- The change in objective function value from $\mathbf{x}^t$ to $\mathbf{x}^t + \lambda \mathbf{d}^y$ ($\lambda \geq 0$) is

$\Rightarrow$ We can use reduced costs to compute changes in objective function

- Suppose we solve a canonical form maximization LP with decision variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ using the simplex method, and end up with:

$$\mathbf{x}^t = (0, 150, 0, 200, 50) \qquad \mathcal{B}^t = \{x_2, x_4, x_5\}$$

$$\mathbf{d}^{x_1} = \left(1, -\frac{1}{2}, 0, -\frac{3}{2}, -\frac{1}{2}\right) \qquad \mathbf{d}^{x_3} = \left(0, -\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}\right)$$

$$\bar{c}_{x_1} = 0 \qquad \bar{c}_{x_3} = -25$$

- Is $\mathbf{x}^t$ optimal?
- Are there multiple optimal solutions?
  - Because the reduced cost $\bar{c}_{x_1} = 0$,
  - Let's explore using x_1 as an entering variable:
- In general, if there is a reduced cost equal to 0 at an optimal solution, there may be other optimal solutions
  - The zero reduced cost must correspond to a simplex direction with $\lambda_{\max} > 0$