

Lesson 25. Weak and Strong Duality

1 Practice taking duals!

Example 1. State the dual of the following linear program.

$$\begin{aligned} \text{maximize} \quad & 5x_1 + x_2 - 4x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 19 \\ & 4x_2 + 8x_4 \leq 55 \\ & x_1 + 6x_2 - x_3 \geq 7 \\ & x_1 \text{ free, } x_2 \geq 0, x_3 \geq 0, x_4 \leq 0 \end{aligned}$$

2 Weak duality

- Let [max] and [min] be a primal-dual pair of LPs
 - [max] is the maximization LP
 - [min] is the minimization LP
 - [min] is the dual of [max], and [max] is the dual of [min]
- Let z^* be the optimal value of [max]
- In the previous lesson, we saw that
 - Any feasible solution to [max] gives a lower bound on z^*
 - Any feasible solution to [min] gives an upper bound on z^*
- Putting these observations together:

Weak duality theorem.

$$\left(\begin{array}{c} \text{Objective function value} \\ \text{of any feasible solution to [max]} \end{array} \right) \leq \left(\begin{array}{c} \text{Objective function value} \\ \text{of any feasible solution to [min]} \end{array} \right)$$

- The weak duality theorem has several interesting consequences

Corollary 1. If \mathbf{x}^* is a feasible solution to [max], \mathbf{y}^* is a feasible solution to [min], and

$$\left(\begin{array}{c} \text{Objective function value} \\ \text{of } \mathbf{x}^* \text{ in [max]} \end{array} \right) = \left(\begin{array}{c} \text{Objective function value} \\ \text{of } \mathbf{y}^* \text{ in [min]} \end{array} \right)$$

then (i) \mathbf{x}^* is an optimal solution to [max], and (ii) \mathbf{y}^* is an optimal solution to [min].

Proof. • Let's start with (i):

$$\left(\begin{array}{c} \text{Objective function value} \\ \text{of } \mathbf{x}^* \text{ in [max]} \end{array} \right) = \left(\begin{array}{c} \text{Objective function value} \\ \text{of } \mathbf{y}^* \text{ in [min]} \end{array} \right) \geq \left(\begin{array}{c} \text{Objective function value} \\ \text{of any feasible solution to [max]} \end{array} \right)$$

- Therefore \mathbf{x}^* must be an optimal solution to [max]
- (ii) can be argued similarly

Corollary 2. (i) If [max] is unbounded, then [min] must be infeasible.
(ii) If [min] is unbounded, then [max] must be infeasible.

Proof. • Let's start with (i)

- Proof by contradiction: suppose [min] is feasible, and let \mathbf{y}^* be a feasible solution to [min]

$$\left(\begin{array}{c} \text{Objective function value} \\ \text{of } \mathbf{y}^* \text{ in [min]} \end{array} \right) \geq \left(\begin{array}{c} \text{Objective function value} \\ \text{of any feasible solution to [max]} \end{array} \right)$$

- Therefore [max] cannot be unbounded, which is a contradiction
- (ii) can be argued similarly
- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
 - See Rader p. 328 for an example

3 Strong duality

Strong duality theorem. Let [P] denote a primal LP and [D] its dual.

- a. If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.
 - b. If [P] and [D] both have feasible solutions, then
 - [P] has a finite optimal solution \mathbf{x}^* ;
 - [D] has a finite optimal solution \mathbf{y}^* ;
 - the optimal values of [P] and [D] are equal.
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- This is an AMAZING fact
 - Useful from theoretical, algorithmic, and modeling perspectives
 - Even the simplex method implicitly uses duality: the reduced costs are essentially solutions to the dual that are infeasible until the last step