Lesson 27. Maximin and Minimax Objectives

1 The minimum of a collection of functions

Example 1. Santa Claus is trying to decide how to give candy canes to three children: Ann, Bob, and Carol. Because Santa is a very busy person, he has decided to give the same number of candy canes to each child. Let x be the number of candy canes each child receives. Also, because Santa knows everything, he knows the happiness level of each child as a function of the number of candy canes he or she receives:

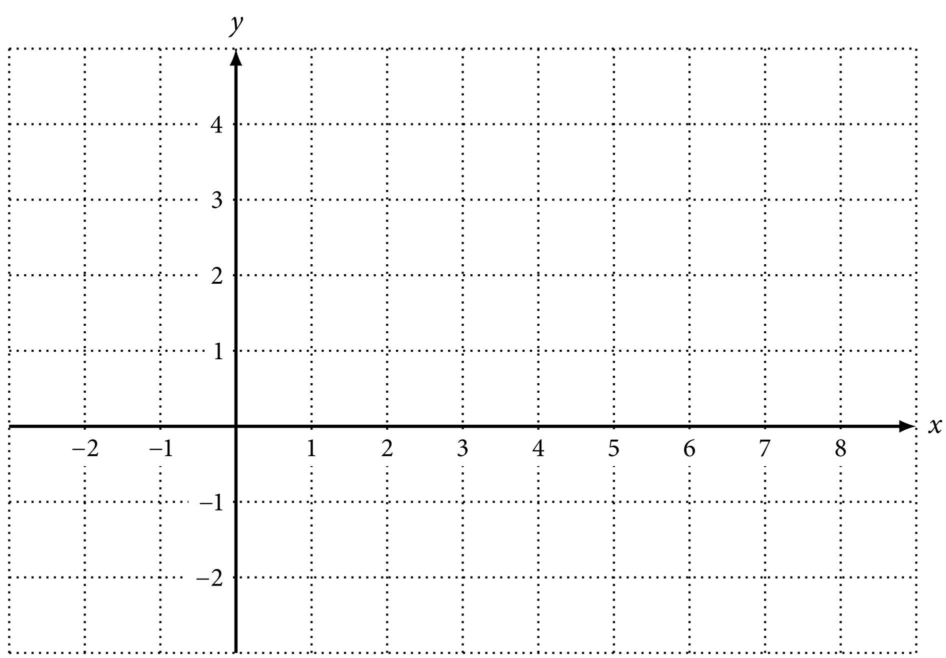
$$\text{Ann:} \quad 1 + 2x \qquad \text{Bob:} \quad 2 + x \qquad \text{Carol:} \quad 5 - \frac{1}{2}x$$

Due to the struggling economy, Santa's budget limits him to give each child at most 6 candy canes. To be fair to all 3 children, he has decided that he wants to **maximize the minimum happiness level of all 3 children**. In other words, he is trying to maximize the worst-case happiness level.

Let $f(x)$ be the minimum happiness level of all 3 children when each child receives x candy canes:

What is $f(0)$? $f(1)$? $f(2)$?

Graph $f(x)$:



Santa's optimization problem is:

By looking at the graph of $f(x)$, give an optimal solution to Santa's optimization problem. What are Ann's, Bob's, and Carol's happiness levels at this solution?

The minimum of a collection of numbers is the largest value that is less than or equal to each number in the collection.

For example, consider $\min\{3, 8, -2, 6, 9\}$.

Using this observation, we can rewrite Santa's optimization problem as:

This looks familiar...

What if we maximized the sum of the happiness factors of all 3 children? What is the optimal solution? What is Ann's, Bob's, and Carol's happiness levels at this solution?

$\Longrightarrow$ **Maximizing the minimum results in more uniform performance than maximizing the sum**

2 Maximin objective functions

- Define:
  - decision variables x_j for $j \in \{1, \dots, n\}$
  - constants a_{ij} for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$
  - constants b_i for $i \in \{1, \dots, m\}$
- Consider an optimization model with a **maximin objective function**:

$$\begin{aligned} \text{maximize} \quad & \min \left\{ \sum_{j=1}^{n} a_{1j} x_j + b_1, \ \sum_{j=1}^{n} a_{2j} x_j + b_2, \ \dots, \ \sum_{j=1}^{n} a_{mj} x_j + b_m \right\} \\ \text{subject to} \quad & \text{(some constraints)} \end{aligned}$$

- We can convert this model into a linear program:
  - Add auxiliary decision variable z
  - Change objective and add constraints:

$$\begin{aligned} \text{maximize} \quad & z \\ \text{subject to} \quad & z \le \sum_{j=1}^{n} a_{ij} x_j + b_i \qquad \text{for } i \in \{1, \dots, m\} \\ & \text{(some constraints)} \end{aligned}$$

- Main idea: the minimum of a collection of numbers is <u>the largest value that is less than or equal to each number in the collection</u>.

3 Minimax objective functions

- We can similarly convert an optimization model with a **minimax objective function**

$$\begin{aligned} \text{minimize} \quad & \max \left\{ \sum_{j=1}^{n} a_{1j} x_j + b_1, \ \sum_{j=1}^{n} a_{2j} x_j + b_2, \ \dots, \ \sum_{j=1}^{n} a_{mj} x_j + b_m \right\} \\ \text{subject to} \quad & \text{(some constraints)} \end{aligned}$$

  into a linear program:
  - Add auxiliary decision variable z
  - Change objective and add constraints:

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & z \ge \sum_{j=1}^{n} a_{ij} x_j + b_i \qquad \text{for } i \in \{1, \dots, m\} \\ & \text{(some constraints)} \end{aligned}$$

- Similar idea: the maximum of a collection of numbers is <u>the smallest value that is greater than or equal to each number in the collection</u>.

Example 2. The State of Simplex wants to divide the effort of its on-duty officers among 8 highway segments to reduce speeding incidents. You, the analyst, were able to estimate that for each highway segment $j \in \{1, \dots, 8\}$, the weekly reduction in speeding incidents is $r_j + s_j x_j$, where x_j is the number of officers assigned to segment j. Due to local ordinances, there is an upper bound u_j on the number of officers assigned to highway segment j per week, for $j \in \{1, \dots, 8\}$. There are 25 officers per week to allocate.

The State of Simplex has decided that it wants to maximize the worst-case reduction in speeding incidents among all highway segments. Write a linear program that allocates officers to highway segments according to this objective.

Input parameters.

$$
\begin{aligned}
H &= \text{set of highway segments} = \{1, \dots, 8\} \\
r_j, s_j &= \text{coefficients on weekly incident reduction function for highway segment } j && \text{for } j \in H \\
u_j &= \text{upper bound on number of officers assigned to highway segment } j \text{ per week} && \text{for } j \in H \\
N &= \text{number of officers per week to allocate} = 25
\end{aligned}
$$

Decision variables.

$$x_j = \text{number of officers assigned to highway segment } j \quad \text{for } j \in H$$

Optimization model with maximin objective function.

$$
\begin{aligned}
\text{maximize} \quad & \min\left\{r_1 + s_1 x_1,\ r_2 + s_2 x_2,\ \dots,\ r_8 + s_8 x_8\right\} \\
\text{subject to} \quad & \sum_{j \in H} x_j = N \\
& x_j \le u_j \quad \text{for } j \in H \\
& x_j \ge 0 \quad \text{for } j \in H
\end{aligned}
$$

Equivalent linear program.