SA305 – Linear Programming Spring 2016
Asst. Prof. Nelson Uhan

Lesson 28. LP Duality and Game Theory

This lesson...

- LP duality and two-player zero-sum game theory

Game theory

- Game theory is the mathematical study of strategic interactions, in which an individual's success depends on his/her own choice as well as the choices of others
- We'll look at one type of game, and use LP duality to give us some insight about behavior in these games

Two-player zero-sum games

- Two players make decisions simultaneously
- Payoff depends on joint decisions
- Zero-sum: whatever one person wins, the other person loses
- Examples:
  - Rock-paper-scissors
  - Advertisers competing for market share (gains/losses over existing market share)

Payoff matrices

- 2 players
  - player R (for "row")
  - player C (for "column")
- Player R chooses among m rows (**actions**)
- Player C chooses among n columns
- Example: rock-paper-scissors, $m = 3$, $n = 3$

	Rock	Paper	Scissors
Rock	0	–1	1
Paper	1	0	–1
Scissors	–1	1	0

- This is the **payoff matrix** for player R
- Zero-sum: Player C receives the negative

- Another example: $m = 2$, $n = 3$

	1	2	3
1	−2	1	2
2	2	−1	0

- Suppose Player R chooses row 2, Player C chooses column 1
- What is the payoff of each player?

Pure and mixed strategies

- **Pure strategy**: pick one row (or column) over and over again
- **Mixed strategy**: each player assigns probabilities to each of his/her strategies
- For example:

	1	2	3
1	−2	1	2
2	2	−1	0
3	1	0	−2

- Suppose player R plays all three actions with equal probability
  - Row 1 with probability 1/3
  - Row 2 with probability 1/3
  - Row 3 with probability 1/3
- For example:

	1	2	3	Prob.
1	−2	1	2	1/3
2	2	−1	0	1/3
3	1	0	−2	1/3
Expected payoffs				

- Suppose player R plays all three actions with equal probability

⇒ Can compute **expected payoffs**:
  - If player C plays
    * column 1:
    * column 2:
    * column 3:

Who has the advantage?

- Can we find "optimal" (mixed) strategies for two-player zero-sum games?
- What can player R guarantee in return, regardless of what C chooses?

Player R and payoff lower bounds

- Suppose Player R plays all three actions with equal probability
- With this mixed strategy, R can guarantee a payoff of at least:
- This is a lower bound on the payoff R gets when playing $(1/3, 1/3, 1/3)$

Player C and payoff upper bounds

	1	2	3	Expected payoff (for R)
1	-2	1	2	
2	2	-1	0	
3	1	0	-2	
Prob.	1/3	1/3	1/3	

- Player C's payoff = $-$(Player R's payoff)
- Player C wants to limit Player R's payoff
- Suppose Player C plays all three actions with equal probability
- With this mixed strategy, C can guarantee that R gets a payoff of at most:
- This is an upper bound on the payoff R gets when C plays $(1/3, 1/3, 1/3)$

Let's optimize: Player R's problem

- Want to decide mixed strategy that maximizes guaranteed payoff

⇒ Decision variables:

$$x_i = \text{prob. of choosing action } i \qquad \text{for } i \in \{1, 2, 3\}$$

	1	2	3	Prob
1	-2	1	2	x_1
2	2	-1	0	x_2
3	1	0	-2	x_3

- Optimization model:

- Player R's problem: maximin

- Convert Player R's problem to LP:

Player C's problem

- Want to decide mixed strategy that limits Player R's payoff

⇒ Decision variables:

$$y_i = \text{prob. of choosing action } i \quad \text{for } i \in \{1, 2, 3\}$$

	1	2	3
1	−2	1	2
2	2	−1	0
3	1	0	−2
Prob.	y_1	y_2	y_3

- Optimization model:

- Player C's problem: minimax

- Convert Player C's problem to LP:

Optimal mixed strategy for Player R

	1	2	3	Prob.
1	-2	1	2	7/18
2	2	-1	0	5/18
3	1	0	-2	1/3
Expected payoff	1/9	1/9	1/9	

- Solve Player R's LP

⇒ Optimal mixed strategy for R guarantees that R can get at least:

- This is the maximin payoff

Optimal mixed strategy for Player C

	1	2	3	Expected payoff (for R)
1	-2	1	2	1/9
2	2	-1	0	1/9
3	1	0	-2	1/9
Prob.	1/3	5/9	1/9	

- Solve Player C's LP

⇒ Optimal mixed strategy for C guarantees that C can limit R's payoff to at most:

- This is the minimax payoff
- Note that maximin payoff = minimax payoff – **not** a coincidence

Fundamental Theorem of 2-Player Zero-Sum Games

- $A = m \times n$ payoff matrix for a 2-player zero-sum game
  - a_{ij} = entries of A

Player R's problem:

$$\begin{aligned} z_R^* = \max \ \min & \left\{ \sum_{i=1}^{m} a_{i1} x_i, \ \ldots, \ \sum_{i=1}^{m} a_{in} x_i \right\} \\ \text{s.t.} & \sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0 \quad \text{for } i \in \{1, \ldots, m\} \end{aligned}$$

Player C's problem:

$$\begin{aligned} z_C^* = \min\ & \max\left\{ \sum_{j=1}^{n} a_{1j} y_j,\ \ldots\ ,\ \sum_{j=1}^{n} a_{nj} y_j \right\} \\ \text{s.t.}\ & \sum_{j=1}^{n} y_j = 1 \\ & y_j \geq 0 \quad \text{for } j \in \{1, \ldots, n\} \end{aligned}$$

- Then, $z_R^* = z_C^*$ i.e. **maximin payoff = minimax payoff**
- Why is this remarkable?
  - Think back to example
  - Imagine you are Player R, and you have to announce in advance what your mixed strategy is
  - Intuitively, this seems like a bad idea
  - But, if you play the optimal maximin strategy, you are guaranteed an expected payoff of 1/9
  - And, Player C cannot do anything to prevent this
  - Announcing the strategy beforehand does not cost you in this case
- Why is this true?
  - Player R's LP and Player C's LP form a primal-dual pair
  - Theorem follows immediately from strong duality for LP
  - For example, after some manipulation, it is easy to see that in our game, Player R's LP and Player C's LP are duals of each other

Player R's LP:

$$\begin{aligned} \max\quad & z \\ \text{s.t.}\quad & 2x_1 - 2x_2 - x_3 + z \leq 0 \\ & -x_1 + x_2 + z \leq 0 \\ & -2x_1 + 2x_3 + z \leq 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

Player C's LP:

$$\begin{aligned} \min\quad & w \\ \text{s.t.}\quad & 2y_1 - y_2 - 2y_3 + w \geq 0 \\ & -2y_1 + y_2 + w \geq 0 \\ & -y_1 + 2y_3 + w \geq 0 \\ & y_1 + y_2 + y_3 = 1 \\ & y_1, \quad y_2, \quad y_3 \geq 0 \end{aligned}$$