Lesson 6. Sets, Summations, For Statements

1 Sets

- A **set** is a collections of elements/objects, e.g.

$$S = \{1, 2, 3, 4, 5\} \qquad \text{Fruits} = \{\text{Apple}, \text{Orange}, \text{Pear}\} \tag{1}$$

- "in" symbol:

$$i \in N \quad \Leftrightarrow \quad \text{"element } i \text{ is in the set } N\text{"}$$

- For example:

$3 \in S$ $\qquad$ $\text{Pear} \in \text{Fruits}$

2 Summations

- Summation symbol over sets:

$$\sum_{i \in N} \quad \Leftrightarrow \quad \text{"sum over all elements of } N\text{"}$$

- For example:

$$\sum_{i \in S} i = 1 + 2 + 3 + 4 + 5 \qquad \sum_{j \in \text{Fruits}} \text{Juice}_j = \text{Juice}_{\text{Apple}} + \text{Juice}_{\text{Orange}} + \text{Juice}_{\text{Pear}}$$

- Common shorthand: if $N = \{1, 2, \ldots, n\}$, then

$$\sum_{i \in N} \quad \text{is the same as} \quad \sum_{i \in \{1,2,\ldots,n\}} \quad \text{as well as} \quad \sum_{i=1}^{n}$$

Example 1. Let the sets S and Fruits be defined as above in (1). Write each of the following as compactly as possible using summation notation:

a. $x_{\text{Apple}} + x_{\text{Orange}} + x_{\text{Pear}}$ $\quad x_j$

b. $1y_1 + 2y_2 + 3y_3 + 4y_4 + 5y_5$ $\quad i y_i$

a. $\displaystyle\sum_{j \in \text{Fruits}} x_j$ $\qquad$ b. $\displaystyle\sum_{i \in S} i y_i$

$\displaystyle\sum_{k \in \text{Fruits}} x_k$

3 For statements

- "for" statements over sets:

$$\text{for } i \in N \quad \Leftrightarrow \quad \text{"repeat for each element of } N\text{"}$$

- For example: Fruits = { Apple, orange, Pear }

$$c_j x_1 + d_j x_2 \le b_j \quad \text{for } j \in \text{Fruits} \quad \Leftrightarrow$$

$$\boxed{\begin{aligned} c_{Apple} x_1 + d_{Apple} x_2 &\le b_{Apple} \\ c_{orange} x_1 + d_{orange} x_2 &\le b_{orange} \\ c_{Pear} x_1 + d_{Pear} x_2 &\le b_{Pear} \end{aligned}}$$

- Common shorthand: if $N = \{1, 2, \ldots, n\}$, then

"for $i \in N$" is the same as "for $i \in \{1, 2, \ldots, n\}$" as well as "for $i = 1, 2, \ldots, n$"

- Sometimes we say "for all $i \in N$" instead of "for $i \in N$" also sometimes "$\forall i \in N$"

4 Multiple indices

- Sometimes it may be useful to use decision variables with multiple indices
- Example:
  - Set of hat types: $H = \{A, B, C\}$
  - Set of factories: $F = \{1, 2\}$
  - Each hat type can be be produced at each factory
  - Define decision variables:

$$x_{i,j} = \text{number of type } i \text{ hats produced at factory } j \qquad \text{for } i \in H \text{ and } j \in F \tag{2}$$

  - What decision variables have we just defined? How many are there?

$3 \times 2 = 6$ decision variables: $x_{A,1}$ $x_{A,2}$ $x_{B,1}$ $x_{B,2}$ $x_{C,1}$ $x_{C,2}$

Example 2. Using the decision variables defined in (2), write expressions for

a. Total number of type C hats produced
b. Total number of hats produced at facility 2

Use summation notation if possible.

a. $x_{C,1} + x_{C,2}$

$$= \sum_{j=1}^{2} x_{C,j} = \sum_{j \in F} x_{C,j}$$

b. $x_{A,2} + x_{B,2} + x_{C,2}$

$$= \sum_{i \in H} x_{i,2}$$

- Suppose

$$c_{i,j} = \text{cost of producing one type } i \text{ hat at factory } j \qquad \text{for } i \in H \text{ and } j \in F$$

- If we produce $x_{i,j}$ hats of type i at factory j (for $i \in H$ and $j \in F$), then the total cost is

$$c_{A,1}x_{A,1} + c_{A,2}x_{A,2} + c_{B,1}x_{B,1} + c_{B,2}x_{B,2} + c_{C,1}x_{C,1} + c_{C,2}x_{C,2}$$

$$= \sum_{j\in F} c_{A,j}x_{A,j} + \sum_{j\in F} c_{B,j}x_{B,j} + \sum_{j\in F} c_{C,j}x_{C,j}$$

$$= \underbrace{\sum_{i\in H}\sum_{j\in F}}_{\text{can also be written as } \sum_{i\in H, j\in F}} c_{i,j}x_{i,j}$$

Example 3. Let $M = \{1,2,3\}$ and $N = \{1,2,3,4\}$. Write the following as compactly as possible using summation notation and "for" statements.

$$\left.\begin{aligned} \text{Let } y_1 &= \text{amount of product 1 produced} \\ y_2 &= \text{amount of product 2 produced} \\ y_3 &= \text{amount of product 3 produced} \\ y_4 &= \text{amount of product 4 produced} \end{aligned}\right\} \text{(1)}$$

$$\left.\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 &= b_1 \\ a_{2,1}y_1 + a_{2,2}y_2 + a_{2,3}y_3 + a_{2,4}y_4 &= b_2 \\ a_{3,1}y_1 + a_{3,2}y_2 + a_{3,3}y_3 + a_{3,4}y_4 &= b_3 \end{aligned}\right\} \text{(2)}$$

(1) $\iff$ Let y_i = amount of product i produced for $i \in N$

(2) $\iff$ $a_{j,1}y_1 + a_{j,2}y_2 + a_{j,3}y_3 + a_{j,4}y_4 = b_j$ for $j \in M$

$\iff$ $\sum_{i\in N} a_{j,i}y_i = b_j$ for $j \in M$