Lesson 7. Resource Allocation and Blending Models, Revisited

- In this lesson, we will learn to write optimization models with sets and parameters

Example 1. Farmer Jones decides to supplement her income by baking and selling two types of cakes, chocolate and vanilla. Each chocolate cake sold gives a profit of \$3, and the profit on each vanilla cake sold is \$4. Each chocolate cake uses 4 eggs and 4 pounds of flour, while each vanilla cake uses 2 eggs and 6 pounds of flour. Farmer Jones has 32 eggs and 48 pounds of flour available. Assume all cakes baked are sold, and fractional cakes are OK. Write a linear program that determines how many of each type of cake should Farmer Jones bake in order to maximize her profit.

- Recall that the linear program we wrote for this problem is

C = number of chocolate cakes to bake

V = number of vanilla cakes to bake

$$
\begin{array}{lll}
\text{maximize} & 3C + 4V & \text{(total profit)} \\
\text{subject to} & 4C + 2V \leq 32 & \text{(eggs available)} \\
& 4C + 6V \leq 48 & \text{(flour available)} \\
& C \geq 0, V \geq 0 &
\end{array}
$$

Example 2. Farmer Jones decides to supplement her income by baking and selling cakes. Let K be the set of cake types that she sells. Each cake k sold yields a profit of p_k, for all $k \in K$. Each cake type requires a certain mixture of ingredients. Let I be the set of ingredients that are used. Each type k cake requires a_{ik} units of ingredient i, for all $i \in I$ and $k \in K$. Farmer Jones has b_i units of ingredient i available, for all $i \in I$. Assume all cakes baked are sold, and fractional cakes are OK. Write a linear program that determines how many of each type of cake should Farmer Jones bake in order to maximize her profit.

- Recall that
  - **constants** are numbers that are fixed
  - **parameters** are constants represented by symbols
- What are the sets and parameters in Example 2?

Sets. K = set of cake types

I = set of ingredients

Parameters. p_k = unit profit for cake type k for $k \in K$

a_{ik} = units of ingredient i used in cake type k for $i \in I$, $k \in K$

b_i = units of ingredient i available for $i \in I$

- How do these sets and parameters relate to the constants given in Example 1?

$K = \{c, v\}$ $\quad p_c = 3$ $\quad a_{e,c} = 4$ $\quad a_{e,v} = 2$

$I = \{e, f\}$ $\quad p_v = 4$ $\quad a_{f,c} = 4$ $\quad a_{f,v} = 6$

$b_e = 32$ $\quad b_f = 48$

- Rewrite the linear program for Example 1 using the parameters you defined above.

DVs. x_c = # chocolate cakes to bake
x_v = # vanilla cakes to bake

$$
\begin{aligned}
\max \quad & p_c x_c + p_v x_v \\
\text{s.t.} \quad & a_{e,c} x_c + a_{e,v} x_v \le b_e \\
& a_{f,c} x_c + a_{f,v} x_v \le b_f \\
& x_c \ge 0, \quad x_v \ge 0
\end{aligned}
$$

maximize	$3C + 4V$	(total profit)
subject to	$4C + 2V \le 32$	(eggs available)
	$4C + 6V \le 48$	(flour available)
	$C \ge 0, V \ge 0$	

- Now write a linear program for Example 2, using summation notation and for statements.

DVs. x_k = # type k cakes to bake for $k \in K$

$$
\begin{aligned}
\max \quad & \sum_{k \in K} p_k x_k & & \text{(total profit)} \\
\text{s.t.} \quad & \sum_{k \in K} a_{i,k} x_k \le b_i & \text{for } i \in I & \text{ (ingredient availability)} \\
& x_k \ge 0 & \text{for } k \in K & \text{ (nonnegativity)}
\end{aligned}
$$

- This is a **parameterized optimization model**: an optimization model where at least one constant is a parameter
- The parameters in this model are placeholders for concrete set elements and numerical values
- This model is valid for any problem of the same structure
  - Just need to specify concrete set elements and numerical values for the parameters
  - e.g. specify elements for K and I; numerical values for p_k for $k \in K$, b_i for $i \in I$, and a_{ik} for $i \in I$ and $k \in K$

Example 3. The Hoosier Gasoline Company produces two blends of gasoline, regular and premium, by mixing three different types of oil. Each type of oil comes in barrels and has its own costs and octane ratings, which are given below:

Type	Cost/Barrel	Octane Rating
1	45	93
2	35	90
3	20	87

Premium gasoline must consist of at least 30% Type 1 oil. In addition, the minimum weighted average octane rating and minimum production requirements for each blend are as follows:

Blend	Weighted Average Octane Rating	Demand
Regular	89	15,000 barrels
Premium	91	12,500 barrels

Formulate a linear program that determines how to meet the demand for each blend of gasoline at minimum cost.

- Recall that the linear program we wrote for this problem is

$$R_1 = \text{number of barrels of Type 1 oil used in regular blend}$$
$$P_1 = \text{number of barrels of Type 1 oil used in premium blend}$$
$$R_2, P_2, R_3, P_3 \text{ defined similarly}$$

$$\begin{array}{lll}
\text{minimize} & 45(R_1 + P_1) + 35(R_2 + P_2) + 20(R_3 + P_3) & \text{(total cost)} \\
\text{subject to} & P_1 \geq 0.3(P_1 + P_2 + P_3) & \text{(premium must have at least 30\% Type 1 oil)} \\
& 93R_1 + 90R_2 + 87R_3 \geq 89(R_1 + R_2 + R_3) & \text{(regular octane requirement)} \\
& 93P_1 + 90P_2 + 87P_3 \geq 91(P_1 + P_2 + P_3) & \text{(premium octane requirement)} \\
& R_1 + R_2 + R_3 \geq 15000 & \text{(regular demand)} \\
& P_1 + P_2 + P_3 \geq 12500 & \text{(premium demand)} \\
& R_1 \geq 0, R_2 \geq 0, R_3 \geq 0, & \text{(nonnegativity)} \\
& P_1 \geq 0, P_2 \geq 0, P_3 \geq 0 &
\end{array}$$

- Describe the constants of this problem as sets and parameters.

Sets. B = set of gas blends = $\{R, P\}$
T = set of oil types = $\{1, 2, 3\}$

Parameters. d_j = demand for blend j for $j \in B$
c_i = cost/barrel for oil i for $i \in T$
r_i = octane rating for oil i for $i \in T$
a_j = weighted avg. octane requirement for blend j for $j \in B$

- Write a parameterized linear program for this problem using the sets and parameters you described above.

$c_1 \sum_{j \in B} x_{1j} + c_2 \sum_{j \in B} x_{2j} + c_3 \sum_{j \in B} x_{3j}$

$c_1(x_{1R} + x_{1P}) + c_2(x_{2R} + x_{2P}) + c_3(x_{3R} + x_{3P})$

$$
\begin{array}{ll}
\text{minimize} & 45(R_1 + P_1) + 35(R_2 + P_2) + 20(R_3 + P_3) \\
\text{subject to} & P_1 \geq 0.3(P_1 + P_2 + P_3) \\
& 93R_1 + 90R_2 + 87R_3 \geq 89(R_1 + R_2 + R_3) \\
& 93P_1 + 90P_2 + 87P_3 \geq 91(P_1 + P_2 + P_3) \\
& \overset{x_{1R} + x_{2R} + x_{3R}}{R_1 + R_2 + R_3} \geq 15000\, d_R \rightarrow \sum_{i \in T} x_{iR} \geq d_R \\
& \overset{x_{1P} + x_{2P} + x_{3P}}{P_1 + P_2 + P_3} \geq 12500\, d_P \rightarrow \sum_{i \in T} x_{iP} \geq d_P \\
& R_1 \geq 0, R_2 \geq 0, R_3 \geq 0, \\
& P_1 \geq 0, P_2 \geq 0, P_3 \geq 0
\end{array}
$$

DVs. $x_{i,j}$ = # barrels of oil $i \rightarrow$ blend j for $i \in T$, $j \in B$

$$\min \quad \sum_{i \in T} c_i \sum_{j \in B} x_{ij} \qquad \text{(total cost)}$$

s.t.

$$x_{1P} \geq 0.3 \sum_{i \in T} x_{iP} \qquad \text{(premium has} \geq 30\% \text{ Type 1 oil)}$$

$$\sum_{i \in T} x_{ij} \geq d_j \quad \text{for } j \in B \qquad \text{(demand requirements)}$$

$$\sum_{i \in T} r_i x_{ij} \geq a_j \sum_{i \in T} x_{ij} \quad \text{for } j \in B \qquad \text{(octane requirements)}$$

$$x_{ij} \geq 0 \quad \text{for } i \in T,\ j \in B \qquad \text{(nonnegativity)}$$