Lesson 11. Graphical Solution of Linear Programs

0 Warm up

Example 1.

a. On the axes on page 2, draw the following equations, and label the points of intersection.

$$4C + 2V = 32$$
$$4C + 6V = 48$$

b. On the same axes, draw the equation $3C + 4V = 18$. Suppose you change 18 to another value. How would your answer change?

1 Overview

- Previously, we formulated a linear program for Farmer Jones's problem:

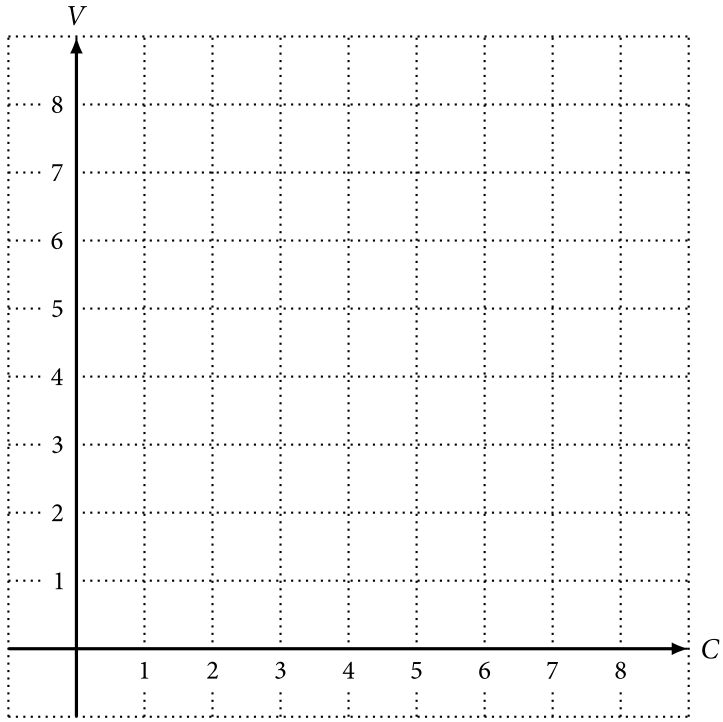
$$C = \text{number of chocolate cakes to bake}$$
$$V = \text{number of vanilla cakes to bake}$$

$$\begin{array}{lll} \text{maximize} & 3C + 4V & \text{(total profit)} \\ \text{subject to} & 4C + 2V \leq 32 & \text{(eggs used vs. available)} \\ & 4C + 6V \leq 48 & \text{(flour used vs. available)} \\ & C \geq 0 & \text{(nonnegativity)} \\ & V \geq 0 & \text{(nonnegativity)} \end{array}$$

- By trial-and-error, the best feasible solution we found was $C = 6$, $V = 4$ with value 34
- Let's find an optimal solution and the optimal value to Farmer Jones's model in a systematic way

2 Solving Farmer Jones's model graphically

- We can graphically solve linear programs with 2 variables
- The feasible region – the collection of all feasible solutions – for Farmer Jones's optimization model:



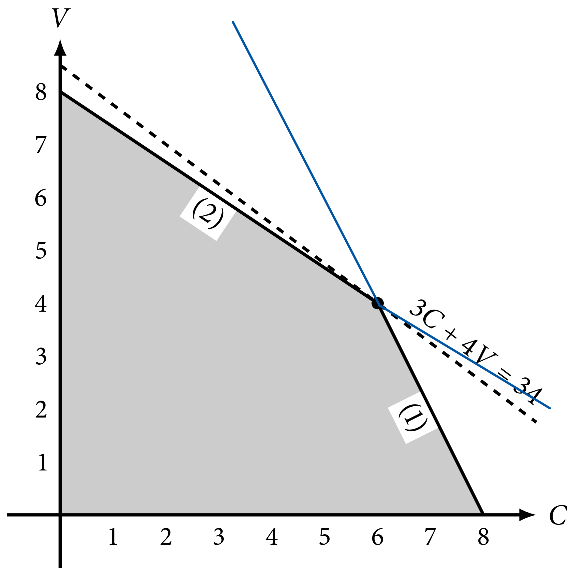
- Any point in this shaded region represents a feasible solution
- How do we find the one with the highest value?
- $C = 6$, $V = 0$ is a feasible solution with value $3(6) + 4(0) = 18$
- The set of (C, V) with value 18 satisfies: $3C + 4V = 18$
- The set of feasible solutions with a value of 18 is graphically represented by:

  the intersection of the line $3C + 4V = 18$ with the feasible region.

- Idea:
  - Draw lines of the form $3C + 4V = k$ for different values of k
  - Find the largest value of k such that the line $3C + 4V = k$ intersects the feasible region
- These lines are called **contour plots**
  - Lines through points having equal objective function value

3 Sensitivity analysis

- For what profit margins on vanilla cakes will the current optimal solution remain optimal?



- Key observation:

The slope of the objective function contour must be "between" the slopes of (1) and (2).

- Slope of (1) = -2, slope of (2) = $-\frac{2}{3}$

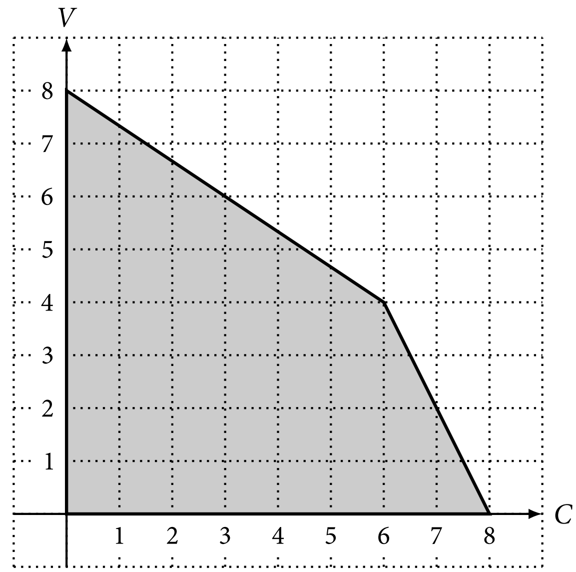
- Let a be the new profit margin on vanilla cakes

  $\Rightarrow$ objective function is $3C + aV$, slope of contour plots = $-\frac{3}{a}$

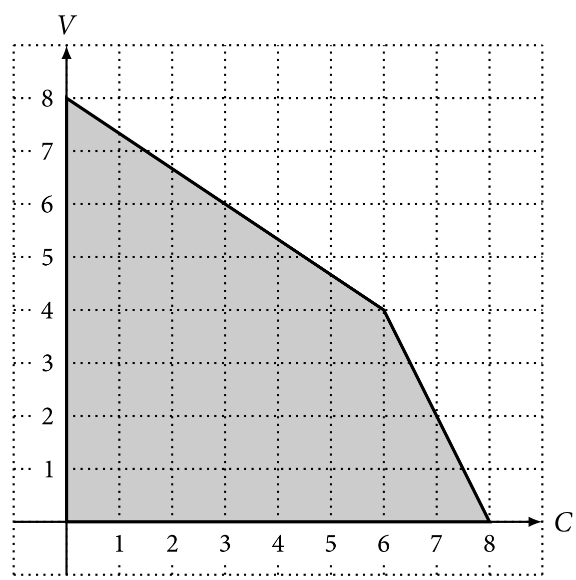
$\Rightarrow$ If $-2 \leq -\frac{3}{a} \leq -\frac{2}{3}$, then the current optimal solution remains optimal

4 Outcomes of optimization models

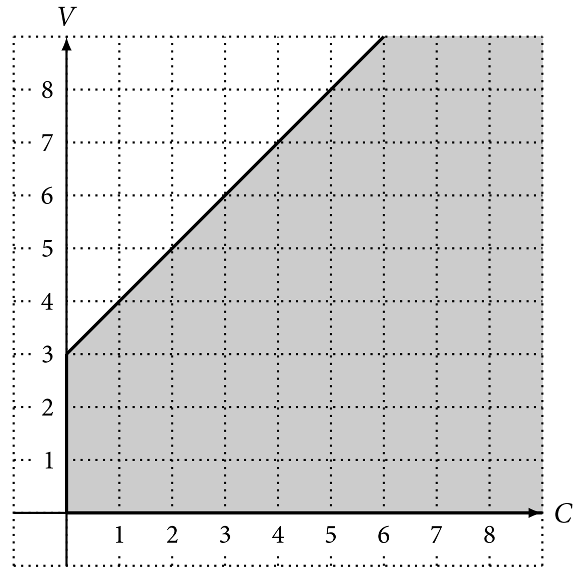
- An optimization model may:
  1. have a **unique optimal solution**
     - e.g. the original Farmer Jones model
  2. have **multiple optimal solutions**
     - e.g. What if the profit margin on chocolate and vanilla cakes is \$2 and \$3, respectively, instead?
     - Farmer Jones's objective function is then $\boxed{2C + 3V}$ → contours have slope $-\frac{2}{3}$



  3. be **infeasible**: no choice of decision variables satisfies all constraints
     - e.g. What if the demands of Farmer Jones's neighbors dictate that he needs to bake at least 9 chocolate cakes?
     - Then we need to add the constraint $\boxed{C \geq 9}$



4. be **unbounded**: for any feasible solution, there exists another feasible solution with a better value
   - e.g. What if the circumstances have changed so that the feasible region of Farmer Jones's model actually looks like this:



5 Next...

- How can we solve linear programs with more than 2 variables?
- Algorithm design
- Improving search and the simplex method