Lesson 16. Linear Programs in Canonical Form

0 Warm up

Example 1.

Let $A = \underbrace{\begin{pmatrix} 1 & 9 & 8 \\ 5 & 2 & 3 \end{pmatrix}}_{2\times 3}$ and $\mathbf{x} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{3\times 1}$. Then $A\mathbf{x} = \boxed{\begin{pmatrix} x_1 + 9x_2 + 8x_3 \\ 5x_1 + 2x_2 + 3x_3 \end{pmatrix}}$.

1 Canonical form

- LP in **canonical form** with decision variables $x_1, \dots, x_n$:

$$\begin{aligned} \text{minimize / maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad \text{for } i \in \{1, \dots, m\} \\ & x_j \geq 0 \qquad \text{for } j \in \{1, \dots, n\} \end{aligned}$$

all general constraints are equalities

all variables are nonnegative

- In vector-matrix notation with decision variable vector $\mathbf{x} = (x_1, \dots, x_n)$:

$$\begin{aligned} \text{minimize / maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{CF}$$

  - A has m rows and n columns, $\mathbf{b}$ has m components, and $\mathbf{c}$ and $\mathbf{x}$ each have n components

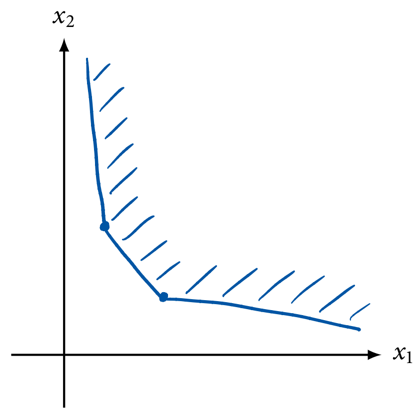
- We typically assume that $m \leq n$, and $\text{rank}(A) = m$

Example 2. Identify $\mathbf{x}$, $\mathbf{c}$, A, and $\mathbf{b}$ in the following canonical form LP:

$$\begin{aligned} \text{maximize} \quad & 3x + 4y - z \\ \text{subject to} \quad & 2x - 3y + z = 10 \\ & 7x + 2y - 8z = 5 \\ & x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{c} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & -3 & 1 \\ 7 & 2 & -8 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

- A canonical form LP always has at least 1 extreme point (if it has a feasible solution)
  - Intuition: if solutions in the feasible region must satisfy $\mathbf{x} \geq \mathbf{0}$, then the feasible region must be "pointed"



2 Converting any LP to an equivalent canonical form LP

- Inequalities → equalities
  - **Slack** and **surplus** variables "consume the difference" between the LHS and RHS
  - If constraint i is a $\leq$-constraint, add a slack variable s_i:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \Rightarrow \quad \boxed{\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i \qquad s_i \geq 0}$$

  - If constraint i is a $\geq$-constraint, subtract a surplus variable s_i:

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \Rightarrow \quad \boxed{\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \qquad s_i \geq 0}$$

- Nonpositive variables → nonnegative variables
  - If $x_j \leq 0$, then introduce a new variable x'_j and substitute $x_j = -x'_j$ everywhere – in particular:

$$\boxed{x_j \leq 0 \quad \Rightarrow \quad -x'_j \leq 0 \quad \Rightarrow \quad x'_j \geq 0}$$

- Unrestricted ("free") variables → nonnegative variables
  - If x_j is unrestricted in sign, introduce 2 new nonnegative variables x_j^+, x_j^-
  - Substitute $x_j = x_j^+ - x_j^-$ everywhere
  - Why does this work?
    - Any real number can be expressed as the difference of two nonnegative numbers

$5 = 5 - 0$
$-4 = 0 - 4 = 1 - 5$

Example 3. Convert the following LPs to canonical form.

(a)
$$\begin{array}{ll} \text{maximize} & 3x + 8y \\ \text{subject to} & x + 4y \leq 20 \\ & x + y \geq 9 \\ & x \geq 0,\ y \text{ free} \end{array}$$

$y = y^+ - y^-$

(b)
$$\begin{array}{ll} \text{minimize} & 5x_1 - 2x_2 + 9x_3 \\ \text{subject to} & 3x_1 + x_2 + 4x_3 = 8 \\ & 2x_1 + 7x_2 - 6x_3 \leq 4 \\ & x_1 \leq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

$x_1 = -x_1'$

(a)
$$\begin{array}{ll} \max & 3x + 8y^+ - 8y^- \\ \text{s.t.} & x + 4y^+ - 4y^- + s_1 = 20 \\ & x + y^+ - y^- - s_2 = 9 \\ & x \geq 0,\ y^+ \geq 0,\ y^- \geq 0,\ s_1 \geq 0,\ s_2 \geq 0 \end{array}$$

(b)
$$\begin{array}{ll} \min & -5x_1' - 2x_2 + 9x_3 \\ \text{s.t.} & -3x_1' + x_2 + 4x_3 = 8 \\ & -2x_1' + 7x_2 - 6x_3 + s_1 = 4 \\ & x_1' \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ s_1 \geq 0 \end{array}$$

3 Basic solutions in canonical form LPs

- Recall: a solution **x** of an LP with n decision variables is a **basic solution** if
  (a) it satisfies all equality constraints
  (b) at least n constraints are active at **x** and are linearly independent
- The solution **x** is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of the LP
- What do basic solutions in canonical form LPs look like?

3.1 Example

- Consider the following canonical form LP:

$$
\begin{array}{lrll}
\text{maximize} & 3x + 8y & & \\
\text{subject to} & x + 4y + s_1 & = 20 & (1) \\
& x + y + s_2 & = 9 & (2) \\
& 2x + 3y + s_3 & = 20 & (3) \\
& x & \geq 0 & (4) \\
& y & \geq 0 & (5) \\
& s_1 & \geq 0 & (6) \\
& s_2 & \geq 0 & (7) \\
& s_3 & \geq 0 & (8)
\end{array}
$$

Canonical form LP:

$$
\begin{array}{rl}
\text{minimize / maximize} & \mathbf{c}^\top \mathbf{x} \\
\text{subject to} & A\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{array}
$$

- Identify the matrix A and the vectors **c**, **x**, and **b** in the above canonical form LP.

$$\vec{x} = \begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} 3 \\ 8 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 20 \\ 9 \\ 20 \end{pmatrix}$$

$m = 3$ $\qquad n = 5$

- Suppose **x** is a basic solution
  - How many linearly independent constraints must be active at **x**? n 5
  - How many of these must be equality constraints? m 3
  - How many of these must be nonnegativity bounds? $n - m$ $5 - 3 = 2$

$$
\begin{array}{lllllll}
\text{maximize} & 3x + 8y & & & & & \\
\text{subject to} & x + 4y + s_1 & & & = & 20 & (1) \\
 & x + \ \ y & + s_2 & & = & 9 & (2) \\
 & 2x + 3y & & + s_3 & = & 20 & (3) \\
 & x & & & \geq & 0 & (4) \\
 & \quad y & & & \geq & 0 & (5) \\
 & \quad\quad s_1 & & & \geq & 0 & (6) \\
 & & s_2 & & \geq & 0 & (7) \\
 & & & s_3 & \geq & 0 & (8)
\end{array}
$$

- Let's compute the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8)
  - It turns out that the constraints (1), (2), (3), (6), and (8) are linearly independent
  - Since the basic solution is active at the nonnegativity bounds (6) and (8),

s_1 and s_3 are "forced" to be 0

  - The other variables, x, y, and s_2 are potentially nonzero
  - Substituting $s_1 = 0$ and $s_3 = 0$ into the other constraints (1), (2), and (3), we get

$$
\begin{aligned}
x + 4y + (0) \quad\quad\quad\quad &= 20 \\
x + \ \ y \quad\quad + s_2 \quad\quad &= 9 \\
2x + 3y \quad\quad\quad\quad + (0) &= 20
\end{aligned}
\tag{$*$}
$$

  - Let $\mathbf{x}_B = (x, y, s_2)$ and B be the submatrix of A consisting of columns corresponding to x, y, and s_2:

$$
B = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix}
$$

  - Note that $(*)$ can be written as

$$
B\mathbf{x}_B = \mathbf{b} \tag{$**$}
$$

  - The columns of B linearly independent. Why?

$\det(B) = \det\begin{pmatrix} 1 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix} = 5 \neq 0 \Rightarrow$ The columns of B are LI

  - $(**)$ has a unique solution. Why?

The columns of B are LI $\Rightarrow$ B is invertible

$\Rightarrow$ $(**)$ has a unique solution: $\vec{x}_B = B^{-1}\vec{b}$

  - It turns out that the solution to $(**)$ is $\mathbf{x}_B = (4, 4, 1)$
  - Put it together: the basic solution $\mathbf{x} = (x, y, s_1, s_2, s_3)$ associated with (1), (2), (3), (6), and (8) is

$$
\left.\begin{array}{l} s_1 = 0, \quad s_3 = 0 \\ \vec{x}_B = (x, y, s_2) = (4, 4, 1) \end{array}\right\} \Rightarrow \vec{x} = (4, 4, 0, 1, 0)
$$

4 Generalizing the example

- Now let's generalize what happened in the example above
- Consider the generic canonical form LP (CF)
  - Let n = number of decision variables
  - Let m = number of equality constraints
  - In other words, A has m rows and n columns
  - Assume $m \leq n$ and $\text{rank}(A) = m$
- Suppose $\mathbf{x}$ is a basic solution
  - How many linearly independent constraints must be active at $\mathbf{x}$? $\boxed{n}$
  - Since $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$, how many nonnegativity bounds must be active? $\boxed{n - m}$
- Generalizing our observations from the example, we have the following theorem:

Theorem 1. If $\mathbf{x}$ is a basic solution of a canonical form LP, then there exists m **basic variables** of $\mathbf{x}$ such that

(a) the columns of A corresponding to these m variables are linearly independent;
(b) the other $n - m$ **nonbasic variables** are equal to 0.

The set of basic variables is referred to as the **basis** of $\mathbf{x}$.

$$
\begin{array}{llrcl}
\text{maximize} & 3x + 8y & & & \\
\text{subject to} & x + 4y + s_1 & = & 20 & (1) \\
& x + y \quad + s_2 & = & 9 & (2) \\
& 2x + 3y \quad + s_3 & = & 20 & (3) \\
& x & \geq & 0 & (4) \\
& y & \geq & 0 & (5) \\
& s_1 & \geq & 0 & (6) \\
& s_2 & \geq & 0 & (7) \\
& s_3 & \geq & 0 & (8)
\end{array}
$$

- Let's check our understanding of this theorem with the example
  - Back in the example, n = $\boxed{5}$ and m = $\boxed{3}$
  - Recall that $\mathbf{x} = (x, y, s_1, s_2, s_3) = (4, 4, 0, 1, 0)$ is a basic solution
  - Which variables of $\mathbf{x}$ correspond to m LI columns of A? $\boxed{x, y, s_2}$
  - Which $n - m$ variables of $\mathbf{x}$ are equal to 0? $\boxed{s_1, s_3}$
  - The basic variables of $\mathbf{x}$ are $\boxed{x, y, s_2}$
  - The nonbasic variables of $\mathbf{x}$ are $\boxed{s_1, s_3}$
  - The basis of $\mathbf{x}$ is $\boxed{\mathcal{B} = \{x, y, s_2\}}$
- Let B be the submatrix of A consisting of columns corresponding to the m basic variables
- Let $\mathbf{x}_B$ be the vector of these m basic variables
- Since the columns of B are linearly independent, the system $B\mathbf{x}_B = \mathbf{b}$ has a unique solution
  - This matches what we saw in ($**$) in the above example
- The m basic variables are potentially nonzero, while the other $n - m$ nonbasic variables are forced to be zero