Lesson 17. The Simplex Method

0 Review

- Given an LP with n decision variables, a solution $\mathbf{x}$ is **basic** if:
  (a) it satisfies all equality constraints
  (b) at least n linearly independent constraints are active at $\mathbf{x}$
- A **basic feasible solution (BFS)** is a basic solution that satisfies all constraints of the LP
- **Canonical form LP**:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

  - m equality constraints and n decision variables (e.g. A has m rows and n columns).
  - Standard assumptions: $m \leq n$, $\text{rank}(A) = m$
- If $\mathbf{x}$ is a basic solution of a canonical form LP, there exist m **basic variables** of $\mathbf{x}$ such that
  (a) the columns of A corresponding to these m variables are linearly independent
  (b) the other $n - m$ **nonbasic variables** are equal to 0
- The set of basic variables is the **basis** of $\mathbf{x}$

1 Overview

- General improving search algorithm

1: Find an initial feasible solution $\mathbf{x}^0$
2: Set $t = 0$
3: **while** $\mathbf{x}^t$ is not locally optimal **do**
4: Determine a simultaneously improving and feasible direction $\mathbf{d}$ at $\mathbf{x}^t$
5: Determine step size λ
6: Compute new feasible solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda \mathbf{d}$
7: Set $t = t + 1$
8: **end while**

- The **simplex method** is a specialized version of improving search
  - For canonical form LPs
  - Start at a BFS in Step 1
  - Consider directions that point towards other BFSes in Step 4
  - Take the maximum possible step size in Step 5

Example 1. Throughout this lesson, we will use the canonical form LP below:

$$\begin{array}{rrcrcrcrcrcl} \text{maximize} & 13x & + & 5y & & & & & & & & \\ \text{subject to} & 4x & + & y & + & s_1 & & & & & = & 24 \\ & x & + & 3y & & & + & s_2 & & & = & 24 \\ & 3x & + & 2y & & & & & + & s_3 & = & 23 \\ & x, & & y, & & s_1, & & s_2, & & s_3 & \geq & 0 \end{array}$$

2 Initial solutions

- For now, we will start by guessing an initial BFS

Example 2. Verify that $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ is a BFS with basis $\mathcal{B}^0 = \{s_1, s_2, s_3\}$.

3 Finding feasible directions

- Two BFSes are **adjacent** if their bases differ by exactly 1 variable
- Suppose $\mathbf{x}^t$ is the current BFS with basis $\mathcal{B}^t$
- Approach: consider directions that point towards BFSes adjacent to $\mathbf{x}^t$
- To get a BFS adjacent to $\mathbf{x}^t$:
  - Put one nonbasic variable into $\mathcal{B}^t$
  - Take one basic variable out of $\mathcal{B}^t$
- Suppose we want to put nonbasic variable y into $\mathcal{B}^t$
- This corresponds to the **simplex direction** $\mathbf{d}^y$ corresponding to nonbasic variable y

- $\mathbf{d}^y$ has a component for every decision variable
  - e.g. $\mathbf{d}^y = (d^y_x, d^y_y, d^y_{s_1}, d^y_{s_2}, d^y_{s_3})$ for the LP in Example 1
- The components of the simplex direction $\mathbf{d}^y$ corresponding to nonbasic variable y are:
  - $d^y_y = 1$
  - $d^y_z = 0$ for all other nonbasic variables z
  - d^y_w (uniquely) determined by $A\mathbf{d} = \mathbf{0}$ for all basic variables w
- Why does this work? Remember for LPs, $\mathbf{d}$ is a feasible direction at $\mathbf{x}$ if
  - $\mathbf{a}^\top \mathbf{d} \leq 0$ for each active constraint of the form $\mathbf{a}^\top \mathbf{x} \leq b$
  - $\mathbf{a}^\top \mathbf{d} \geq 0$ for each active constraint of the form $\mathbf{a}^\top \mathbf{x} \geq b$
  - $\mathbf{a}^\top \mathbf{d} = 0$ for each active constraint of the form $\mathbf{a}^\top \mathbf{x} = b$
- Each nonbasic variable has a corresponding simplex direction

Example 3. The basis of the BFS $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ is $\mathcal{B}^0 = \{s_1, s_2, s_3\}$. For each nonbasic variable, x and y, we have a corresponding simplex direction. Compute the simplex directions $\mathbf{d}^x$ and $\mathbf{d}^y$.

4 Finding improving directions

- Once we've computed the simplex direction for each nonbasic variable, which one do we choose?
- We choose a simplex direction $\mathbf{d}$ that is improving
- Recall that if $f(\mathbf{x})$ is the objective function, $\mathbf{d}$ is an improving direction at $\mathbf{x}$ if

$$\nabla f(\mathbf{x})^\top \mathbf{d} \begin{cases} > 0 & \text{when maximizing } f \\ < 0 & \text{when minimizing } f \end{cases}$$

- For LPs, $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$, and so $\nabla f(\mathbf{x}) =$ ________ for any $\mathbf{x}$
- The **reduced cost** associated with nonbasic variable y is

$$\bar{c}_y = \mathbf{c}^\top \mathbf{d}^y$$

  where $\mathbf{d}^y$ is the simplex direction associated with y

- The simplex direction $\mathbf{d}^y$ associated with nonbasic variable y is improving if

$$\bar{c}_y \begin{cases} > 0 & \text{for a maximization LP} \\ < 0 & \text{for a minimization LP} \end{cases}$$

Example 4. Consider the BFS $\mathbf{x}^0 = (0, 0, 24, 24, 23)$ with basis $\mathcal{B}^0 = \{s_1, s_2, s_3\}$. Compute the reduced costs $\bar{c}_x$ and $\bar{c}_y$ for nonbasic variables x and y, respectively. Are $\mathbf{d}^x$ and $\mathbf{d}^y$ improving?

- If there is an improving simplex direction, we choose it
- If there is more than 1 improving simplex direction, we can choose any one of them
  - One option – **Dantzig's rule**: choose the improving simplex direction with the most improving reduced cost (maximization LP – most positive, minimization LP – most negative)
- **If there are no improving simplex directions, then the current BFS is a global optimal solution**

5 Determining the maximum step size

- We've picked an improving simplex direction – how far can we go in that direction?
- Suppose $\mathbf{x}^t$ is our current BFS, $\mathbf{d}$ is the improving simplex direction we chose
- Our next solution is $\mathbf{x}^t + \lambda\mathbf{d}$ for some value of $\lambda \geq 0$
- How big can we make λ while still remaining feasible?
- Recall that we computed $\mathbf{d}$ so that $A\mathbf{d} = \mathbf{0}$
- $\mathbf{x}^t + \lambda\mathbf{d}$ satisfies the equality constraints $A\mathbf{x} = \mathbf{b}$ no matter how large λ gets, since

$$A(\mathbf{x}^t + \lambda\mathbf{d}) = A\mathbf{x}^t + \lambda A\mathbf{d} = A\mathbf{x}^t = \mathbf{b}$$

- So, the only thing that can go wrong are the nonnegativity constraints

  $\Rightarrow$ What is the largest λ such that $\mathbf{x}^t + \lambda\mathbf{d} \geq \mathbf{0}$?

Example 5. Suppose we choose the improving simplex direction $\mathbf{d}^x = (1, 0, -4, -1, -3)$. Compute the maximum step size λ for which $\mathbf{x}^0 + \lambda \mathbf{d}^x$ remains feasible.

- Note that only negative components of $\mathbf{d}$ determine maximum step size:

$$x_j + \lambda d_j \overset{?}{\geq} 0$$

- The **minimum ratio test**: starting at the BFS $\mathbf{x}$, if any component of the improving simplex direction $\mathbf{d}$ is negative, then the maximum step size is

$$\lambda_{\max} = \min \left\{ \frac{x_j}{-d_j} : d_j < 0 \right\}$$

Example 6. Verify that the minimum ratio test yields the same maximum step size you found in Example 5.

- What if $\mathbf{d}$ has no negative components?
- For example:
  - Suppose $\mathbf{x}^0 = (0, 0, 1, 2, 3)$ is a BFS
  - $\mathbf{d} = (1, 0, 2, 4, 3)$ is an improving simplex direction at $\mathbf{x}$
  - Then the next solution is

$$\mathbf{x}^0 + \lambda \mathbf{d} = (\lambda, 0, 1 + 2\lambda, 2 + 4\lambda, 3 + 3\lambda) \quad \text{for some value of } \lambda \geq 0$$

  - $\mathbf{x}^0 + \lambda \mathbf{d} \geq 0$ for all $\lambda \geq 0$!

- We can improve our objective function and remain feasible forever!

⇒ The LP is unbounded

- **Test for unbounded LPs**: if all components of an improving simplex direction are nonnegative, then the LP is unbounded

6 Updating the basis

- We have our improving simplex direction $\mathbf{d}$ and step size $\lambda_{\max}$
- We can compute our new solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max}\mathbf{d}$
- We also update the basis: update the set of basic variables
- **Entering and leaving variables**
  - The nonbasic variable corresponding to the chosen simplex direction enters the basis and becomes basic: this is the **entering variable**
  - Any one of the basic variables that define the maximum step size leaves the basis and becomes nonbasic: this is the **leaving variable**

Example 7. Compute $\mathbf{x}^1$. What is the basis $\mathcal{B}^1$ of $\mathbf{x}^1$?

7 Putting it all together: the simplex method

Step 0: Initialization. Identify a BFS $\mathbf{x}^0$. Set solution index $t = 0$.

Step 1: Simplex directions. For each nonbasic variable y, compute the corresponding simplex direction $\mathbf{d}^y$ and its reduced cost $\bar{c}_y$.

Step 2: Check for optimality. If no simplex direction is improving, stop. The current solution $\mathbf{x}^t$ is optimal. Otherwise, choose any improving simplex direction $\mathbf{d}$. Let x_e denote the entering variable.

Step 3: Step size. If $\mathbf{d} \geq \mathbf{0}$, stop. The LP is unbounded. Otherwise, choose the leaving variable x_ℓ by computing the maximum step size $\lambda_{\max}$ according to the minimum ratio test.

Step 4: Update solution and basis. Compute the new solution $\mathbf{x}^{t+1} = \mathbf{x}^t + \lambda_{\max}\mathbf{d}$. Replace x_ℓ with x_e in the basis. Set $t = t + 1$. Go to Step 1.

Problem 1. Consider the following LP

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 \leq 18 \\ & 4x_1 + 2x_2 + 5x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{1}$$

The canonical form of this LP is

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 + s_1 = 18 \\ & 4x_1 + 2x_2 + 5x_3 + s_2 = 10 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned} \tag{2}$$

a. Use the simplex method to solve the canonical form LP (2). In particular:

- Use the initial BFS $\mathbf{x}^0 = (0, 0, 0, 18, 10)$ with basis $\mathcal{B}^0 = \{s_1, s_2\}$.
- Choose your entering variable using **Dantzig's rule** – that is, choose the improving simplex direction with the most positive reduced cost. (If this was a minimization LP, you would choose the improving simplex direction with the most negative reduced cost.)

b. What is the optimal value of the canonical form LP (2)? Give an optimal solution.

c. What is the optimal value of the original LP (1)? Give an optimal solution.