Lesson 18. Finding an Initial BFS

1 Overview

- Today: How do we find an initial BFS to start the simplex method?
- The **Phase I LP**: an auxiliary LP based on the original canonical form LP with an easy-to-find initial BFS
  - Solve the Phase I LP using the simplex method
  - The optimal solution to the Phase I LP will either
    - give an initial BFS for the original LP
    - prove that the original LP is infeasible

2 Constructing the Phase I LP

1. If necessary, multiply the equality constraints by −1 so that the RHS is nonnegative
2. Add a nonnegative **artificial variable** to the LHS of each equality constraint
   (each equality constraint gets its own artificial variable)
3. The objective is to minimize the sum of the artificial variables
4. Compute the initial BFS for the Phase I LP by putting all artificial variables in the basis

Example 1. Construct the Phase I LP from the following canonical form LP.

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 5x_2 - 9x_3 \\ \text{subject to} \quad & 8x_1 - x_2 + x_3 = 4 \\ & x_1 + 4x_2 - 7x_3 = -22 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned} \qquad (*)$$

$n = 3$
$m = 2$

What is the initial BFS of the Phase I LP?

Phase I LP:

$$\begin{aligned} \min \quad & a_1 + a_2 \\ \text{s.t.} \quad & 8x_1 - x_2 + x_3 + a_1 = 4 \\ & -x_1 - 4x_2 + 7x_3 + a_2 = 22 \\ & x_1, x_2, x_3, a_1, a_2 \geq 0 \end{aligned}$$

$m = 2$
$n = 5$

Initial BFS for Phase I LP: $\mathcal{B} = \{a_1, a_2\}$

$$\vec{x}^0 = (\underbrace{0, 0, 0}_{\text{nonbasic}}, \underbrace{4, 22}_{\text{basic}})$$

3 How does the Phase I LP work?

- Let's consider the Phase I LP we wrote in Example 1
- The Phase I LP can't be unbounded, because $\boxed{a_1 \geq 0, \quad a_2 \geq 0 \quad \Rightarrow \quad a_1 + a_2 \geq 0}$
- It can't be infeasible either (we can always compute an initial BFS!)
- Therefore, the Phase I LP must have an optimal solution
- Let $(x_1^*, x_2^*, x_3^*, a_1^*, a_2^*)$ be an optimal BFS to the Phase I LP
- **Case 1.** The optimal value of the Phase I LP is strictly greater than 0: $a_1^* + a_2^* > 0$

$\Rightarrow$ Any feasible solution to Phase I LP has $a_1 + a_2 \geq a_1^* + a_2^* > 0$

$\Rightarrow$ In any feasible solution to the Phase I LP, either $a_1 > 0$ or $a_2 > 0$ (or both)

$\Rightarrow$ $(*)$ has no feasible solutions!

- **Case 2.** The optimal value of the Phase I LP is equal to 0: $a_1^* + a_2^* = 0$

$\Rightarrow$ $a_1^* = 0$, $a_2^* = 0$ $\Rightarrow$ (x_1^*, x_2^*, x_3^*) is a feasible solution to $(*)$

$(x_1^*, x_2^*, x_3^*, a_1^*, a_2^*)$ is a BFS to Phase I LP

$\Rightarrow$ at least $5 - 2 = 3$ of these components must be equal to 0

$\Rightarrow$ (x_1^*, x_2^*, x_3^*) has at least $3 - 2 = 1$ component equal to 0

$\Rightarrow$ (x_1^*, x_2^*, x_3^*) is a BFS for $(*)$

- This reasoning applies in general

4 Putting it all together: The Two-Phase Simplex Method

Step 1: Phase I. Construct Phase I LP and compute its easy-to-find initial BFS. Use the simplex method to solve the Phase I LP.

Step 2: Infeasibility. If the optimal value of the Phase I LP is

- $> 0 \quad \Rightarrow$ stop; original LP is infeasible.
- $= 0 \quad \Rightarrow$ identify initial BFS for original LP.

Step 3: Phase II. Use the simplex method to solve the original LP, using the initial BFS identified in Step 2.

5 Possible outcomes of LPs

- When do we detect if an LP:

is infeasible?	is unbounded?	has an optimal solution?
Phase I	Phase II	Phase II