Lesson 19. Degeneracy, Convergence, Multiple Optimal Solutions

0 Warm up

Example 1. Suppose we are using the simplex method to solve the following canonical form LP:

$\vec{c} = (10, 3, 0, 0, 0)$

$$
\begin{array}{llr}
\text{maximize} & 10x + 3y & \\
\text{subject to} & x + y + s_1 = 4 & (1) \\
& 5x + 2y + s_2 = 11 & (2) \\
& y + s_3 = 4 & (3) \\
& x \geq 0 & (4) \\
& y \geq 0 & (5) \\
& s_1 \geq 0 & (6) \\
& s_2 \geq 0 & (7) \\
& s_3 \geq 0 & (8)
\end{array}
$$

$n = 5$
$m = 3$
$\Rightarrow n - m = 2$

Let $\mathbf{x} = (x, y, s_1, s_2, s_3)$. Our current BFS is $\mathbf{x}^t = (0, 4, 0, 3, 0)$ with basis $\mathcal{B}^t = \{y, s_1, s_2\}$. The simplex directions are $\mathbf{d}^x = (1, 0, -1, -5, 0)$ and $\mathbf{d}^{s_3} = (0, -1, 1, 2, 1)$. Compute $\mathbf{x}^{t+1}$ and $\mathcal{B}^{t+1}$.

$\bar{c}_x = \vec{c}^T \vec{d}^x = 10$

$\bar{c}_{s_3} = \vec{c}^T \vec{d}^{s_3} = -3$

$\Rightarrow \vec{d}^x$ is improving, since $\bar{c}_x > 0$ and this is a maximizing LP

$\Rightarrow x$ is entering

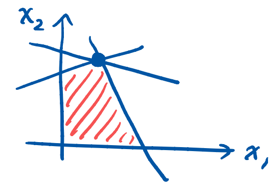
MRT: $\lambda_{max} = \min \left\{ \underset{s_1}{\frac{0}{-(-1)}}, \underset{s_2}{\frac{3}{-(-5)}} \right\} = 0$ $\Rightarrow s_1$ is leaving

$\Rightarrow \vec{x}^{t+1} = \vec{x}^t + \lambda_{max} \vec{d}^x$

$= (0, 4, 0, 3, 0) + 0\,(1, 0, -1, -5, 0) = (0, 4, 0, 3, 0)$

$\mathcal{B}^{t+1} = \{x, y, s_2\}$

- In the above example, the step size $\lambda_{\max} = 0$
- As a result, $\mathbf{x}^{t+1} = \mathbf{x}^t$: it looks like our solution didn't change!
- The basis did change, however: $\mathcal{B}^{t+1} \neq \mathcal{B}^t$
- Why did this happen?



1 Degeneracy

- A BFS $\mathbf{x}$ of an LP with n decision variables is **degenerate** if there are more than n constraints active at $\mathbf{x}$
  - i.e. there are multiple collections of n linearly independent constraints that define the same $\mathbf{x}$

> **Example 2.** Is $\mathbf{x}^t$ in Example 1 degenerate? Why?
>
> Yes. $\vec{x}^t$ is active at $\underbrace{(1), (2), (3)}_{\text{equality constraints}}, \underbrace{(4), (6), (8)}_{\text{nonnegativity constraints.}}$

$\mathcal{B}^t = \{y, s_1, s_2\}$

- In $\mathbf{x}^t = (0, 4, 0, 3, 0)$ in Example 1, "too many" of the nonnegativity constraints are active
  - As a result, some of the basic variables are equal to zero
- Recall: a BFS of a canonical form LP with n decision variables and m equality constraints has
  - $\boxed{m}$ basic variables, potentially zero or nonzero
  - $\boxed{n - m}$ nonbasic variables, always equal to 0

$Ax = b \;]\, m$
$x \geq 0 \;]\, n$

- Suppose $\mathbf{x}$ is a degenerate BFS, with $n + k$ active constraints ($k \geq 1$)
- Then $\boxed{n + k - m}$ nonnegativity bounds must be active, which is larger than $n - m$
- Therefore: a BFS $\mathbf{x}$ of a canonical form LP is degenerate if

  $\boxed{\text{at least one of the basic variables is equal to 0.}}$

- As a result, a degenerate BFS may correspond to several bases
  - e.g. in Example 1, the BFS $(0, 4, 0, 3, 0)$ has bases: $\boxed{\{y, s_1, s_2\}, \; \{x, y, s_2\}}$
- Every step of the simplex method
  - does not necessarily move to a geometrically adjacent extreme point
  - does move to an adjacent BFS (in particular, the bases differ by exactly 1 variable)
- At a degenerate BFS, the simplex method might "get stuck" for a few steps
  - Same BFS, different bases, different simplex directions
  - Zero-length moves: $\lambda_{\max} = 0$
- When $\lambda_{\max} = 0$, just proceed as usual
- Simplex computations will normally escape a sequence of zero-length moves and move away from the current BFS

2 Convergence

- In extreme cases, degeneracy can cause the simplex method to cycle over a set of bases that all represent the same extreme point
    - See Rader p. 291 for an example
- Can we guarantee that the simplex method terminates?
- Yes! Anticycling rules exist
- Easy anticycling rule: **Bland's rule**
    - Fix an ordering of the decision variables and rename them so that they have a common index
        - e.g. $(x, y, s_1, s_2, s_3) \to (x_1, x_2, x_3, x_4, x_5)$
    - Entering variable: choose nonbasic variable with <u>smallest index</u> among those corresponding to improving simplex directions
    - Leaving variable: choose basic variable with <u>smallest index</u> among those that define $\lambda_{\max}$

3 Multiple optimal solutions

- Suppose our current BFS is $\mathbf{x}^t$, and y is the entering variable
- The change in objective function value from $\mathbf{x}^t$ to $\mathbf{x}^t + \lambda \mathbf{d}^y$ ($\lambda \geq 0$) is

$$\vec{c}^T(\vec{x}^t + \lambda \vec{d}^y) - \vec{c}^T\vec{x}^t = \cancel{\vec{c}^T\vec{x}^t} + \lambda \vec{c}^T\vec{d}^y - \cancel{\vec{c}^T\vec{x}^t} = \lambda \vec{c}^T\vec{d}^y = \lambda \bar{c}_y$$

⇒ We can use reduced costs to compute changes in objective function

- Suppose we solve a canonical form maximization LP with decision variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ using the simplex method, and end up with:

$$\mathbf{x}^t = (0, 150, 0, 200, 50) \qquad \mathcal{B}^t = \{x_2, x_4, x_5\}$$
$$\mathbf{d}^{x_1} = \left(1, -\frac{1}{2}, 0, -\frac{3}{2}, -\frac{1}{2}\right) \qquad \mathbf{d}^{x_3} = \left(0, -\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}\right)$$
$$\bar{c}_{x_1} = 0 \qquad \bar{c}_{x_3} = -25$$

- Is $\mathbf{x}^t$ optimal?

Yes, there are no improving simplex directions

- Are there multiple optimal solutions?
  - Because the reduced cost $\bar{c}_{x_1} = 0$,

    $\vec{d}^{x_1}$ points towards other solutions w/ the same value as $\vec{x}^t$

  - Let's explore using x_1 as an entering variable:

MRT: $\lambda_{max} = \min\left\{ \underset{x_2}{\frac{150}{-(-\frac{1}{2})}}, \underset{x_4}{\frac{200}{-(-\frac{3}{2})}}, \underset{x_5}{\frac{50}{-(-\frac{1}{2})}} \right\} = 100$ $\quad x_5$ leaving

$$\Rightarrow \vec{x}^{t+1} = \vec{x}^t + \lambda_{max}\vec{d}^{x_1} = (0, 150, 0, 200, 50) + 100\left(1, -\tfrac{1}{2}, 0, -\tfrac{3}{2}, -\tfrac{1}{2}\right)$$
$$= (100, 100, 0, 50, 0) \qquad \mathcal{B}^{t+1} = \{x_1, x_2, x_4\}$$

⇒ $\vec{x}^{t+1}$ has the same value as $\vec{x}^t$ ⇒ $\vec{x}^{t+1}$ is also optimal!

- In general, if there is a reduced cost equal to 0 at an optimal solution, there may be other optimal solutions
  - The zero reduced cost must correspond to a simplex direction with $\lambda_{\max} > 0$