Lesson 20. Bounds and the Dual LP

1 Overview

- It is often useful to quickly generate lower and upper bounds on the optimal value of an LP
- Many algorithms for optimization problems that consider LP "subproblems" rely on this
- How can we do this?

2 Finding lower bounds

Example 1. Consider the following LP:

$$\begin{aligned} z^* = \text{maximize} \quad & 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 5x_3 \leq 18 && (1) \\ & 5x_1 + 4x_2 + 3x_3 \leq 16 && (2) \\ & x_1, x_2, x_3 \geq 0 && (3) \end{aligned}$$

Denote the optimal value of this LP by z^*. Give a feasible solution to this LP and its value. How does this value compare to z^*?

Feasible Solution	Value
$(1, 1, 1)$	9
$(1, 2, 1)$	12
$(0, 2, 2)$	14

each of these values is $\leq z^*$

- **For a maximization LP, any feasible solution gives a lower bound on the optimal value**
- We want the highest lower bound possible (i.e. the lower bound closest to the optimal value)

3 Finding upper bounds

- We want the lowest upper bound possible (i.e. the upper bound closest to the optimal value)
- For the LP in Example 1, we can show that the optimal value z^* is at most 27
  - Any feasible solution (x_1, x_2, x_3) must satisfy constraint (1)

  $\Rightarrow$ Any feasible solution (x_1, x_2, x_3) must also satisfy constraint (1) multiplied by 3/2 on both sides:

$$\frac{3}{2}(3x_1 + 2x_2 + 5x_3) \leq \frac{3}{2}(18)$$
$$\Leftrightarrow \frac{9}{2}x_1 + 3x_2 + \frac{15}{2}x_3 \leq 27$$

  - The nonnegativity bounds (3) imply that any feasible solution (x_1, x_2, x_3) must satisfy

$$2x_1 + 3x_2 + 4x_3 \leq \frac{9}{2}x_1 + 3x_2 + \frac{15}{2}x_3 \leq 27$$

  - Therefore, any feasible solution, including the optimal solution, must have value at most 27
- We can do better: we can show $z^* \leq 25$:
  - Any feasible solution (x_1, x_2, x_3) must satisfy constraints (1) and (2)

  $\Rightarrow$ Any feasible solution (x_1, x_2, x_3) must also satisfy $\left(\frac{1}{2} \times \text{constraint (1)}\right) + \text{constraint (2)}$:

$$\frac{1}{2}(3x_1 + 2x_2 + 5x_3) + (5x_1 + 4x_2 + 3x_3) \leq \frac{1}{2}(18) + 16$$
$$\Leftrightarrow \frac{13}{2}x_1 + 5x_2 + \frac{11}{2}x_3 \leq 25$$

  - The nonnegativity bounds (3) then imply that any feasible solution (x_1, x_2, x_3) must satisfy

$$2x_1 + 3x_2 + 4x_3 \leq \frac{13}{2}x_1 + 5x_2 + \frac{11}{2}x_3 \leq 25$$

Example 2. Combine the constraints (1) and (2) of the LP in Example 1 to find a better upper bound on z^* than 25.

$$\begin{aligned} z^* = \text{maximize} \quad & 2x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 5x_3 \leq 18 \quad (1) \\ & 5x_1 + 4x_2 + 3x_3 \leq 16 \quad (2) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Any feasible solution (x_1, x_2, x_3) must satisfy $\frac{1}{2}(1) + \frac{1}{2}(2)$:

$$\frac{1}{2}(3x_1 + 2x_2 + 5x_3) + \frac{1}{2}(5x_1 + 4x_2 + 3x_3) \leq \frac{1}{2}(18) + \frac{1}{2}(16)$$
$$\Leftrightarrow 4x_1 + 3x_2 + 4x_3 \leq 17$$

(3) $\Rightarrow$ any feasible solution (x_1, x_2, x_3) must satisfy

$$2x_1 + 3x_2 + 4x_3 \leq 4x_1 + 3x_2 + 4x_3 \leq 17 \quad \Rightarrow \quad z^* \leq 17$$

- Let's generalize this process of combining constraints
- Let y_1 be the "multiplier" for constraint (1), and let y_2 be the "multiplier" for constraint (2)
- We require $y_1 \geq 0$ and $y_2 \geq 0$ so that multiplying constraints (1) and (2) by these values keeps the inequalities as "$\leq$"

$$
\begin{array}{rll}
z^* = \text{maximize} & 2x_1 + 3x_2 + 4x_3 & \\
\text{subject to} & 3x_1 + 2x_2 + 5x_3 \leq 18 & (1) \quad y_1 \\
& 5x_1 + 4x_2 + 3x_3 \leq 16 & (2) \quad y_2 \\
& x_1, x_2, x_3 \geq 0 &
\end{array}
$$

- We also want:

$$
\begin{aligned}
3y_1 + 5y_2 &\geq 2 \\
2y_1 + 4y_2 &\geq 3 \\
5y_1 + 3y_2 &\geq 4
\end{aligned}
$$

- Since we want the lowest upper bound, we want:

$$\min \quad 18y_1 + 16y_2$$

- Putting this all together, we can find the multipliers that find the best lower upper bound with the following LP!

$$
\begin{array}{rl}
\text{minimize} & 18y_1 + 16y_2 \\
\text{subject to} & 3y_1 + 5y_2 \geq 2 \\
& 2y_1 + 4y_2 \geq 3 \\
& 5y_1 + 3y_2 \geq 4 \\
& y_1 \geq 0, y_2 \geq 0
\end{array}
$$

  - This is the **dual LP**, or simply the **dual** of the LP in Example 1
  - The LP in example is referred to as the **primal LP** or the **primal** – the original LP

4 In general...

- Every LP has a dual
- For minimization LPs
  - Any feasible solution gives an upper bound on the optimal value
  - One can construct a dual LP to give the greatest lower bound possible
- We can generalize the process we just went through to develop some mechanical rules to construct duals

5 Constructing the dual LP

0. Rewrite the primal so all variables are on the LHS and all constants are on the RHS
1. Assign each primal constraint a corresponding **dual variable** (multiplier)
2. Write the dual objective function
   - The objective function coefficient of a dual variable is the RHS coefficient of its corresponding primal constraint
   - The dual objective sense is the opposite of the primal objective sense
3. Write the dual constraint corresponding to each primal variable
   - The dual constraint LHS is found by looking at the coefficients of the corresponding primal variable ("go down the column")
   - The dual constraint RHS is the objective function coefficient of the corresponding primal variable
4. Use the **SOB rule** to determine dual variable bounds (≥ 0, ≤ 0, free) and dual constraint comparisons ($\leq$, $\geq$, $=$)

	max LP	$\leftrightarrow$	min LP	
sensible	$\leq$ constraint	$\leftrightarrow$	$y_i \geq 0$	sensible
odd	$=$ constraint	$\leftrightarrow$	y_i free	odd
bizarre	$\geq$ constraint	$\leftrightarrow$	$y_i \leq 0$	bizarre
sensible	$x_i \geq 0$	$\leftrightarrow$	$\geq$ constraint	sensible
odd	x_i free	$\leftrightarrow$	$=$ constraint	odd
bizarre	$x_i \leq 0$	$\leftrightarrow$	$\leq$ constraint	bizarre

Example 3. Take the dual of the following LP:

$$\begin{array}{lrcll}
\text{minimize} & 10x_1 + 9x_2 - 6x_3 & & & \\
\text{subject to} & 2x_1 - x_2 & \geq 3 & S & y_1 \\
 & 5x_1 + 3x_2 - x_3 & \leq 14 & B & y_2 \\
 & x_2 + x_3 & = 1 & O & y_3 \\
 & \underset{S}{x_1 \geq 0}, \underset{B}{x_2 \leq 0}, \underset{S}{x_3 \geq 0} & & &
\end{array}$$

Dual:

$$\begin{array}{lrcll}
\max & 3y_1 + 14y_2 + y_3 & & & \\
\text{s.t.} & 2y_1 + 5y_2 + 0y_3 & \leq 10 & S & x_1 \\
 & -y_1 + 3y_2 + y_3 & \geq 9 & B & x_2 \\
 & -y_2 + y_3 & \leq -6 & S & x_3 \\
 & \underset{S}{y_1 \geq 0}, \underset{B}{y_2 \leq 0}, \underset{O}{y_3 \text{ free}} & & &
\end{array}$$

Example 4. Take the dual of the dual LP you found in Example 3.

Dual of the dual:

$$
\begin{array}{llcl}
\min & 10x_1 + 9x_2 - 6x_3 & & \\
\text{s.t.} & 2x_1 - x_2 + 0x_3 \geq 3 & y_1 & S \\
& 5x_1 + 3x_2 - x_3 \leq 14 & y_2 & B \\
& 0x_1 + x_2 + x_3 = 1 & y_3 & O \\
& x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \geq 0 & & \\
& \quad S \qquad\quad B \qquad\quad S & &
\end{array}
$$

$$
\begin{array}{llcl}
\max & 3y_1 + 14y_2 + y_3 & & \\
\text{s.t.} & 2y_1 + 5y_2 + 0y_3 \leq 10 & S & x_1 \\
& -y_1 + 3y_2 + y_3 \geq 9 & B & x_2 \\
& -y_2 + y_3 \leq -6 & S & x_3 \\
& y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \text{ free} & & \\
& \quad S \qquad\quad B \qquad\quad O & &
\end{array}
$$

- In general, **the dual of the dual is the primal**

6 Up next...

- Duality theorems: relationships between the primal and dual LPs