Lesson 21. Weak and Strong Duality

0 Warm up

Example 1. State the dual of the following linear program.

$$
\begin{array}{lllll}
\text{maximize} & 5x_1 + x_2 - 4x_3 & & & \\
\text{subject to} & x_1 + x_2 + x_3 + x_4 = 19 & & O & y_1 \\
& 4x_2 + 8x_4 \leq 55 & & S & y_2 \\
& x_1 + 6x_2 - x_3 \geq 7 & & B & y_3 \\
& x_1 \text{ free}, x_2 \geq 0, x_3 \geq 0, x_4 \leq 0 & & & \\
& \quad O \quad\; S \quad\;\; S \quad\;\; B & & &
\end{array}
$$

$$
\begin{array}{lllll}
\min & 19y_1 + 55y_2 + 7y_3 & & & \\
\text{s.t.} & y_1 \qquad\qquad + y_3 = 5 & & O & x_1 \\
& y_1 + 4y_2 + 6y_3 \geq 1 & & S & x_2 \\
& y_1 \qquad\qquad - y_3 \geq -4 & & S & x_3 \\
& y_1 + 8y_2 \leq 0 & & B & x_4 \\
& \underset{O}{y_1 \text{ free}}, \quad \underset{S}{y_2 \geq 0}, \quad \underset{B}{y_3 \leq 0} & & &
\end{array}
$$

1 Weak duality

- Let [max] and [min] be a primal-dual pair of LPs
  - [max] is the maximization LP
  - [min] is the minimization LP
  - [min] is the dual of [max], and [max] is the dual of [min]
- Let z^* be the optimal value of [max]
- In the previous lesson, we saw that
  - Any feasible solution to [max] gives a lower bound on z^*
  - Any feasible solution to [min] gives an upper bound on z^*
- Putting these observations together:

Weak duality theorem.

$$
\left(\begin{array}{c}\text{Objective function value} \\ \text{of any feasible solution to [max]}\end{array}\right) \leq \left(\begin{array}{c}\text{Objective function value} \\ \text{of any feasible solution to [min]}\end{array}\right)
$$

- The weak duality theorem has several interesting consequences

Corollary 1. If $\mathbf{x}^*$ is a feasible solution to [max], $\mathbf{y}^*$ is a feasible solution to [min], and

$$\begin{pmatrix} \text{Objective function value} \\ \text{of } \mathbf{x}^* \text{ in [max]} \end{pmatrix} = \begin{pmatrix} \text{Objective function value} \\ \text{of } \mathbf{y}^* \text{ in [min]} \end{pmatrix}$$

then (i) $\mathbf{x}^*$ is an optimal solution to [max], and (ii) $\mathbf{y}^*$ is an optimal solution to [min].

Proof.

- Let's start with (i):

$$\begin{pmatrix} \text{Objective function value} \\ \text{of } \mathbf{x}^* \text{ in [max]} \end{pmatrix} = \begin{pmatrix} \text{Objective function value} \\ \text{of } \mathbf{y}^* \text{ in [min]} \end{pmatrix} \underset{\uparrow \atop \text{weak duality}}{\geq} \begin{pmatrix} \text{Objective function value} \\ \text{of any feasible solution to [max]} \end{pmatrix}$$

- Therefore $\mathbf{x}^*$ must be an optimal solution to [max]
- (ii) can be argued similarly

Corollary 2. (i) If [max] is unbounded, then [min] must be infeasible.
(ii) If [min] is unbounded, then [max] must be infeasible.

Proof.

- Let's start with (i)
- Proof by contradiction: suppose [min] is feasible, and let $\mathbf{y}^*$ be a feasible solution to [min]

$$\begin{pmatrix} \text{Objective function value} \\ \text{of } \mathbf{y}^* \text{ in [min]} \end{pmatrix} \underset{\uparrow \atop \text{weak duality}}{\geq} \begin{pmatrix} \text{Objective function value} \\ \text{of any feasible solution to [max]} \end{pmatrix}$$

- Therefore [max] cannot be unbounded, which is a contradiction
- (ii) can be argued similarly

- Note that primal infeasibility does <u>not</u> imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
  - See Rader p. 328 for an example

2 Strong duality

Strong duality theorem. Let [P] denote a primal LP and [D] its dual.

a. If [P] has a finite optimal solution, then [D] also has a finite optimal solution with the same objective function value.

b. If [P] and [D] both have feasible solutions, then
   - [P] has a finite optimal solution $\mathbf{x}^*$;
   - [D] has a finite optimal solution $\mathbf{y}^*$;
   - the optimal values of [P] and [D] are equal.

- This is an AMAZING fact
- Useful from theoretical, algorithmic, and modeling perspectives
- Even the simplex method implicitly uses duality: the reduced costs are essentially solutions to the dual that are infeasible until the last step