Lesson 22. An Economic Interpretation of LP Duality

1 Overview

- An economic interpretation of duality
- Complementary slackness

2 Warm up

Example 1. The Fulkerson Furniture Company produces desks, tables, and chairs. Each type of furniture requires a certain amount of lumber, finishing, and carpentry:

Resource	Desk	Table	Chair	Available
Lumber (sq ft)	8	6	2	48
Finishing (hrs)	3	2	1	20
Carpentry (hrs)	2	2	1	8
Profit ($)	60	30	20	

Assume that all furniture produced is sold, and that fractional solutions are acceptable. Write a linear program to determine how much furniture Fulkerson should produce in order to maximize its profits.

DVs: x_1 = # desks to produce
x_2 = # tables to produce
x_3 = # chairs to produce

$$
\begin{array}{llll}
\max & 60x_1 + 30x_2 + 20x_3 & & \text{(total profit)} \\
\text{s.t.} & 8x_1 + 6x_2 + 2x_3 \leq 48 & & \text{(lumber)} \\
& 3x_1 + 2x_2 + x_3 \leq 20 & & \text{(finishing)} \\
& 2x_1 + 2x_2 + x_3 \leq 8 & & \text{(carpentry)} \\
& x_1, x_2, x_3 \geq 0 & &
\end{array}
$$

3 Economic interpretation of the dual LP

- Suppose an entrepreneur wants to purchase all of Fulkerson's resources (lumber, finishing, carpentry)
- What prices should she offer for the resources that will entice Fulkerson to sell?

- Define decision variables:

$$y_1 = \text{price of 1 sq. ft. lumber}$$
$$y_2 = \text{price of 1 hour of finishing}$$
$$y_3 = \text{price of 1 hour of carpentry}$$

Resource	Desk	Table	Chair	Available
Lumber (sq ft)	8	6	2	48
Finishing (hrs)	3	2	1	20
Carpentry (hrs)	2	2	1	8
Profit ($)	60	30	20	

- To buy all of Fulkerson's resources, entrepreneur pays:

$$48y_1 + 20y_2 + 8y_3$$

- Entrepreneur wants to minimize this cost
- Entrepreneur also needs to offer resource prices that will entice Fulkerson to sell
- One desk uses
  - 8 sq. ft. of lumber
  - 3 hours of finishing
  - 2 hours of carpentry
- One desk has profit of $60

$\Rightarrow$ Entrepreneur should pay at least $60 for this combination of resources:

$$8y_1 + 3y_2 + 2y_3 \geq 60$$

- One table uses
  - 6 sq. ft. of lumber
  - 2 hours of finishing
  - 2 hours of carpentry
- One table has profit of $30

$\Rightarrow$ Entrepreneur should pay at least $30 for this combination of resources:

$$6y_1 + 2y_2 + 2y_3 \geq 30$$

- One chair uses
  - 2 sq. ft. of lumber
  - 1 hours of finishing
  - 1 hours of carpentry
- One chair has profit of $20

$\Rightarrow$ Entrepreneur should pay at least $20 for this combination of resources:

$$2y_1 + y_2 + y_3 \geq 20$$

- Increasing the availability of the resources potentially increases the maximum profits Fulkerson can achieve

$\Rightarrow$ Entrepreneur should pay nonnegative amounts for each resource:

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0$$

- Putting this all together, we get:

$$\begin{array}{rrcrcrl} \min & 48y_1 & + & 20y_2 & + & 8y_3 & \\ \text{s.t.} & 8y_1 & + & 3y_2 & + & 2y_3 \geq 60 & \qquad (x_1\text{: desks}) \\ & 6y_1 & + & 2y_2 & + & 2y_3 \geq 30 & \qquad (x_2\text{: tables}) \\ & 2y_1 & + & y_2 & + & y_3 \geq 20 & \qquad (x_3\text{: chairs}) \\ & y_1, & & y_2, & & y_3 \geq 0 & \end{array}$$

- This is the dual of Fulkerson's LP!
- In summary:
  - Optimal dual solution $\Leftrightarrow$ "fair" prices for associated resources
  - Known as **marginal prices** or **shadow prices**
- Strong duality $\Rightarrow$

$$\left(\begin{array}{c} \text{Company's maximum revenue} \\ \text{from selling furniture} \end{array} \right) = \left(\begin{array}{c} \text{Entrepreneur's minimum cost} \\ \text{of purchasing resources} \end{array} \right)$$

  - Equilibrium under perfect competition: company makes no excess profits
- This kind of economic interpretation is trickier for LPs with different types of constraints and variable bounds

4 Complementary slackness

- Optimal solution to Fulkerson's LP: $x_1 = 4, \; x_2 = 0, \; x_3 = 0$
- Resources used:

  lumber: $32 < 48$ finishing: $12 < 20$ carpentry: $8 = 8$

- How much would you pay for an extra sq. ft. of lumber? $y_1 = 0$
- How much would you pay for an extra hour of finishing? $y_2 = 0$
- Resource not fully utilized in optimal solution

  $\Rightarrow$ marginal price = 0

- **Primal complementary slackness:** either
  - a primal constraint is active at a primal optimal solution, or
  - the corresponding dual variable at optimality = 0

- Same logic applies to the dual
- Dual constraints ⇔ Primal decision variables
- **Dual complementary slackness:** either
  - a primal decision variable at optimality = 0, or
  - the corresponding dual constraint is active in a dual optimal solution

5 More duality practice

Example 2. Consider the following LP:

$$
\begin{array}{lll}
\text{minimize} & 3x_1 - x_2 + 8x_3 & \\
\text{subject to} & -x_1 + 8x_3 \leq 6 & B \quad y_1 \\
 & 5x_1 - 3x_2 + 9x_3 \geq -2 & S \quad y_2 \\
 & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \geq 0 & \\
 & \quad S \qquad\ B \qquad\ S &
\end{array}
$$

	max LP	↔	min LP	
sensible	$\leq$ constraint	↔	$y_i \geq 0$	sensible
odd	= constraint	↔	y_i free	odd
bizarre	$\geq$ constraint	↔	$y_i \leq 0$	bizarre
sensible	$x_i \geq 0$	↔	$\geq$ constraint	sensible
odd	x_i free	↔	= constraint	odd
bizarre	$x_i \leq 0$	↔	$\leq$ constraint	bizarre

a. Write the dual.
b. Find a feasible solution to the primal and the dual.
c. Give a lower and an upper bound on the optimal value of the above LP.

a.

$$
\begin{array}{lllll}
\max & 6y_1 - 2y_2 & & & \\
\text{s.t.} & -y_1 + 5y_2 \leq 3 & & S & x_1 \\
 & -3y_2 \geq -1 & & B & x_2 \\
 & 8y_1 + 9y_2 \leq 8 & & S & x_3 \\
 & y_1 \leq 0,\ y_2 \geq 0 & & & \\
 & \ B \qquad\quad S & & &
\end{array}
$$

b. primal: $(0, 0, 0)$
value $= 0$

dual: $(0, 0)$
value $= 0$

dual (max) primal (min)

c. $0 \leq \text{opt. value} \leq 0$

$\Rightarrow \text{opt. value} = 0$